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16. Abstract

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Sustainable Intersection Control Accommodating Urban Freight Mobility

by

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Research Report SWUTC/09/476660-00015-1

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ABSTRACT

In this report, we studied green extension of a two-phased vehicle actuated signal at an isolated intersection between two one-way streets. The green phase is extended by a preset time interval, referred to as critical gap, from the time of a vehicle arrival. The green phase switches if there is no arrival during the critical gap. We developed a model following a Poisson process that studied the intersection performance of traffic. We extended the model to the case of general traffic. Additionally, we derived the system performance measures. Our findings show that our model is flexible in accommodating freight traffic and that our model in the general case is asymptotically accurate under heavy traffic. Numerical tests show that presence of critical gaps increases vehicle delay in most cases. This finding is enlightening regarding current practices.

EXECUTIVE SUMMARY

Traffic engineers always believe that appropriate design of signals can effectively improve the urban mobility for both passenger and commercial vehicles. However, the complexity in signal control and traffic behavior often makes it difficult to understand the effects of control policies. Although many theoretical papers have conducted analyses of signal control strategies, a clear view on how critical factors determine signal efficiency is still lacking. The objectives of this study are to systematically examine the performance of a popular vehicle-actuated signal control. We chose to study an intersection between two one-way streets. We hope through this special case to gain insights into the properties of general actuated signal systems.

This study was sponsored by the Southwest Regional University Transportation Center (SWUTC) at Texas A&M University from September 1, 2008, through August 31, 2009. Dr. Bruce Wang was the principal investigator of this research. Some of the results herein are from Dr. Wang's research prior to this project.

In this research, we studied a particular operational strategy for the actuated signal system. An advance detector is located at a certain distance prior to an intersection such that an arriving vehicle triggers a green time extension after the queue has been discharged. The extended green period actuated by the vehicle is called critical gap in this report. If there is no vehicle actuation during the critical gap, the green phase switches to clear queues in other approaches. In this way, the actuated system dynamically allocates green time according to vehicle arrivals from multiple approaches. The control strategy therefore reduces to determining the green extension.

Vehicle waiting time at an intersection is a critical performance measure of intersection control. Because of the uncertainty of traffic arrivals, the average waiting time for vehicles is used. The average waiting time is a function of several signal and intersection characteristics, such as the expectation and the variance of green time, which themselves also serve as performance measures. In considering environmental impact, the number of stop-and-goes is used as an alternative performance measure as opposed to the average waiting time. Developing stochastic

models is necessary for the performance because of their capability of accommodating the stochastic traffic.

The optimal solution is developed for the case in which traffic follows Poisson arrival processes. However, when the traffic follows a general stochastic process, such as a general renewal process, an approximated method is needed for performance analysis. Therefore, approximated solutions have been developed for the case of general but heavy traffic. In the case of heavy traffic, the results are asymptotically optimal.

We have found in almost all cases that the optimal control is to switch the signal as soon as the queue had been cleared in one direction. This strategy leads to minimized average vehicle waiting time, regardless of consideration of environmental emissions, with both homogeneous and heterogeneous traffic. The research team believes that this finding can be generalized to the actuated signal system with more than two approaches. In practice, detection of queue existence could adopt various technologies. If vehicle headway is used for this purpose, it would be impossible to reduce the critical gap to zero. In this case, the critical gap is set to an appropriate small value. This practical convenience, however, takes a toll on the system performance in most cases, according to our findings in this research.

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CHAPTER 1:
INTRODUCTION

BACKGROUND

Vehicle-actuated signal systems have been widely adopted in traffic control for their adaptability to traffic arrivals in urban areas. Vehicle mobility, including that of commercial vehicles, and transportation system sustainability are closely related to design of the control strategies at these intersections. A number of studies and practical applications have concluded that vehicle-actuated intersection control promotes sustainability and urban mobility.

Intersection control has several implications. One is reduction of overall vehicle delay at intersections. This is important for growing urban areas. Second, proper intersection control helps improve environment protection. Vehicle stop-and-go at intersections causes environmental concerns, and appropriate intersection control can minimize the number of stop-and-goes. A third implication is that signal control can adapt to different traffic mixes with vehicles of differing characteristics, such as varying values of time. For instance, commercial vehicles' value of time is significantly different from that of passenger cars. An optimal intersection control strategy should explicitly consider heterogeneity of traffic mix.

This study concerns the optimal structure of intersection control at vehicle-actuated signals. Despite the fact that single systems have been used for decades, the optimal structure of the control policy for a vehicle-actuated intersection has not been clear, especially with heterogeneous vehicle mixes. We address these challenges in this project.

Historical Evolution

There is a long history of vehicle-actuated signals. The first traffic signals appeared in London in the United Kingdom in December, 1868. They resembled railway signals of the time. In the United States, the first electric signals were installed in Cleveland, Ohio, in 1914, for the purpose of allowing police and fire stations to control the signals in case of an emergency (USPTO, 1913). In 1926, the first vehicle-actuated signal (a horn-actuated signal) was made in the United

States by placing microphones on the side of the road. In the 1930s and thereafter, vehicle-actuated signals were constructed. During the 1950s and 1960s, with the improvement of engineering techniques, the vehicle-actuated signal system became more and more popular and attracted the attention of traffic engineering researchers.

Relevant Literature

Research on vehicle-actuated control traces back to the 1940s. Clayton (1941) studied the average delay at vehicle-actuated signals on a minor road with light traffic. Tanner (1953) estimated the delays of two opposing streams of vehicles trying to cross a length of road wide enough for only one vehicle at a time. Based on his assumption, he found explicit results for waiting times for only some special cases.

Darroch *et al*. (1964) gave a complete analysis based on assumptions of general distribution of departure vehicle headways and lost time, and of exponential distributed arrival headways. The authors used results of standard queueing theory to investigate the influence of green extension to signal cycle and average delay. In addition, Newell (1969) was primarily concerned with vehicle-actuated control strategies under the arrival probability invariant to time translations. This model did not assume any specific probability function for the arrivals but a linear relationship between the mean and variance of arrivals within any given time period. Furthermore, he evaluated the mean and variance of the cycle time. An important result is that he showed that the cycle time had a gamma distribution.

Based on a binomial vehicle arrival, Dunne (1967) presented a discrete model following Darroch *et al*. (1964). He gave an analytic expression of the probability generating function for vehicle delay and showed a connection to the discrete cases of Darroch *et al*. (1964). Little (1971) also considered a classic discrete model and obtained probability distribution of queue length in the steady state.

Taking bunching arrivals into consideration, Cowan (1978) also considered vehicle-actuated traffic control. The intersection under his consideration was similar to that of Darroch *et al*. (1964) in that the intersection was between two one-way streets. The difference between the two was that the latter assumed a vehicle detector upstream of the intersection while Cowan (1978) assumed it at the stop line of the intersection. He assumed that the bunch of arrivals, separated by inter-bunch headway of shifted exponential distribution, has a general distribution size and the intra-bunch gap has a constant one unit time. The control policy he considered is that the green phase ceases as soon as there is no departure at an allotted time for a particular backward interval length. Our research here follows the assumption in Cowan (1978).

There are also many other proposed models similar to the ones above. Examples include Morris and Pak-Poy (1967) and Lin (1982). Recently, Viti and Zuylen (2009) developed a computational probability model based on an assumption of any temporal distribution of input. Thus, the temporal evolution of queue length, signal sequence probabilities, delay, and waiting time can be computed. Their model can serve as a probabilistic evaluation for microscopic simulations and is also suitable for planning. However, their computational model does not give theoretical insight to control strategy. Readers cannot obtain an optimal control strategy through their analysis.

On the practical front, Kruger *et al*. (1990) studied the real implementation of vehicle-actuated control at an intersection. They investigated the location of detectors for a specific strategy: queue control. Queue control strategy aims to detect the end of the queue and switches the signal at the same time that the end of the queue is expected to reach the intersection. Based on numerical simulation, they showed that, in general, queue control at an isolated intersection is better than other control strategies. The queue control strategy corresponds to the case in our model that the critical gap is zero. We analytically show why queue control is optimal in most cases.

OBJECTIVES OF THIS STUDY

This study mainly focused on assessment of properties for vehicle-actuated signal timing. In particular, it had the following objectives.

- Examining the stochastic properties of signal timing in this particularly popular signal control scheme when vehicle arrivals follow Poisson processes,

- Identifying optimal signal control strategies in this actuated signal system,
- Examining the impact of freight mobility on the optimal configuration of signal control with the concerns of total delays and environmental impact respectively, and
- Examining the stochastic properties of signal timing with a general arrival distribution under heavy traffic.

Outline of This Report

In this report, we define our problem in Chapter 2. This is followed by a derivation of analytical formulas for the expected green time, as well as variances. Chapter 3 addresses the average vehicle delay in terms of expected green time and variance of green time. Also, numerical solutions and some important implications from the optimal control strategy are presented in this chapter. Chapter 4 reports the actuated control under general arrival pattern. We conclude this report in Chapter 5 by summarizing the major results and highlighting some future challenges.

CHAPTER 2:
EXPECTATION AND VARIANCE OF GREEN TIMES

PROBLEM STATEMENT

Modeling methodologies of actuated intersection control highly depends on configuration of intersections. Sometimes, a slight difference as small as the presence of left turning lanes can pose great challenges in model development or its resultant accuracy. Problem setup therefore appears critical. In choosing an intersection for this study, we have several concerns. First, the intersection should provide general insights to other intersections. Second, the intersection setup should be mathematically maneuverable.

In order to guide the subsequent model development, the study problem is defined as follows[1].

We consider an actuated signal system at an isolated intersection between two one-way streets without turning vehicles, denoted by major and minor approaches respectively. We assume that there are only two green phases, each dedicated to traffic in one approach. Vehicle arrivals from the two approaches follow two independent, homogeneous Poisson processes. We consider a special yet popular control scheme that first of all ensures queue clearance during a green phase. After waiting vehicles are discharged, if there is a vehicle arrival at time t, the green phase is extended until time $t+\Delta$. Δ is called *critical gap*. If, however, there is no vehicle arrival during the green period extended, say, $[t, t+\Delta]$, the green phase switches to its conflict approach automatically, resulting in a loss of effective green time. The critical gaps are denoted by Δ_s and Δ_L for the minor and major approaches, respectively. In each approach, there is a constant discharge rate to clear waiting vehicles. Obviously, the critical gaps Δ_s and Δ_L are the only control variables. The objective is to decide the values of Δ_s and Δ_L such that the average vehicle delay at the intersection over a long period is minimized.

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[1] Similar setups are also seen in Darroch et al. (1964), Newell (1969), and Cowan (1978).

It is clear that the critical gap is the only control variable in this control scheme. Figure 1 in Appendix B illustrates such an intersection. For the ease of understanding, we assume that there is only one lane in each direction.

Note that in this idealized study problem, we do not consider the physical location of the loop detector. We assume that it is determined by the critical gap, or that it is simply located at the intersection. We further assume that there is no minimum green time in each approach, as we assume that there is an available technology to detect the vehicle queue presence. The queue clearance time is therefore assumed known to the system each time. Each time after dissipating the queue, the preset critical gap is applied. This setup allows us to study for the full potential in efficient signal control. In addition, we do not consider the physical sizes of the roadway, the intersection, or the vehicles, which is consistent to the point queue model. For the purpose of presentation, we only consider homogeneous passenger cars in the model. In a later section, we will discuss impact and strategies of commercial vehicles.

Strategy

For simplicity, we assume a *strategy* in which the green phase always switches to its conflict approach when there is no vehicle arrival during the period of a critical gap.

We refer to this strategy as the *always-switch* strategy. This strategy is also assumed in Darroch *et al*. (1964). Although the *always-switch* strategy could give rise to a `peculiar' situation in which the green phase switches to the minor approach without any waiting vehicles, and then switches back to the major approach, there is a high probability of vehicle presence at the time of switch. Therefore, we have reasonable confidence that the analysis under this strategy makes a good approximation to the actual performance.

The *always-switch* strategy could also have practical representations. For example, a mobile server travels between two different locations to serve customers in two different geographic areas. Each time the machine moves, some effective service time is lost. If there is no communication between the two service sites, we have an *always-switch* strategy.

As will be seen, vehicle delay and green times are functions of the total green time loss during a signal cycle, irrespective of its split between the two switches. For simplicity, we denote the total green time loss with δ , which, in another word, is the total time loss for the two switches during one signal cycle.

We present the notation next. Note that the subscripts s and L correspond to minor and major approaches, respectively.

Notation

λ_s	vehicle arrival rate along the minor approach
λ_L	vehicle arrival rate along the major approach
Δ_s	critical gap in the minor approach
Δ_L	critical gap in the major approach
t_s	random green time in the minor approach
t_L	random green time in the major approach
f_s	discharge rate of vehicular queue along the minor approach
f_L	discharge rate of vehicular queue along the major approach
δ	total green time loss in a signal cycle due to phase switches
$E[\ ]$	expected value function
$Var(\ )$	variance function
$COV(\ )$	covariance function
$X(\ )$	random number of arrivals with the parameter being time period.

The parameters of the intersection, such as the green time loss and queue discharge rates, are all considered deterministic in this report. We believe that assuming randomness for them would only increase technical complexity slightly and that it would not dramatically change the nature of the results. In addition, the critical gaps, once set up, do not change.

EXPECTED GREEN TIMES

In each approach, the green time consists of two random components: queue clearance time and free flow time. The queue clearance time represents the period in which the discharging rate of vehicles equals the saturation rate, denoted by t_{sa} and t_{La} for both approaches, respectively. The free flow time, denoted by t_{sb} and t_{Lb} for both approaches respectively, corresponds to the period of time, the length of which is a function of the critical gap and vehicle arrivals. In the free flow time, there is no presence of vehicular queue. Note that the duration of free flow time is a function of the critical gap, while the latter is the endogenous variable to be studied. For instance, if the critical gap is set to be zero, which means the signal switches immediately after queue clearance, the free flow time turns to be zero.

As studied in Wang (2007), t_{sb} and t_{Lb} are determined by an information relay process: $E[t_{sb}] = \frac{1}{\lambda_s} e^{\lambda_s \Delta_s} - \frac{1}{\lambda_s}$ and $E[t_{Lb}] = \frac{1}{\lambda_L} e^{\lambda_L \Delta_L} - \frac{1}{\lambda_L}$. To summarize, we have $t_s = t_{sa} + t_{sb}$ and $t_L = t_{La} + t_{Lb}$. Due to the Markov property, t_{sa} and t_{sb} are independent of each other. The same is true for t_{sb} and t_{Lb}. In Appendix B, Figure 2 shows an illustrative queueing process in the minor direction during a cycle. Next, we show how to evaluate t_{sa}. Since the discharge rate of vehicles at the intersection remains constant, then t_{sa} should be a multiple of f_s. However, for convenient analysis, we can still use the continuous model to evaluate the stochastic properties. Doing this would not make the results different. Therefore, t_{sa} satisfies the following relationship for traffic flow conservation.

$$X(t_L + \delta) + X(t_{sa}) - f_s t_{sa} = 0 \quad (1)$$

where $X(\cdot)$ represents the number of arrivals in a Poisson process along the minor approach. Equation (1) states that the total number of vehicles discharged at the saturation rate in time t_{sa} equals the total arrivals during $t_L + \delta + t_{sa}$. Equation (1) ensures a full discharge rate during t_{sa}.

Taking expectation at both sides of Equation (1) gives the following expected queue clearance time. It is worthy to note that t_{sa} is a function of arrival distribution and departure rate; thus any attempt to take expectation or variance by conditioning on t_{sa} and t_L will lead to incorrect results. Here we can replace t_{sa} and t_L by their expectation first and then take expectation at both sides of Equation (1). In the next section, another method using queueing theory will be provided.

$$E[t_{sa}] = \frac{\lambda_s E[t_L] + \lambda_s \delta}{f_s - \lambda_s}$$

Similarly,

$$E[t_{La}] = \frac{\lambda_L E[t_s] + \lambda_L \delta}{f_L - \lambda_L}$$

Therefore, we have

$$\begin{aligned} E[t_s] &= E[t_{sa}] + E[t_{sb}] \\ &= \frac{\lambda_s E[t_L] + \lambda_s \delta}{f_s - \lambda_s} + \frac{1}{\lambda_s} e^{\lambda_s \Delta_s} - \frac{1}{\lambda_s} \end{aligned} \tag{2}$$

In the same way,

$$\begin{aligned} E[t_L] &= E[t_{La}] + E[t_{Lb}] \\ &= \frac{\lambda_L E[t_s] + \lambda_L \delta}{f_L - \lambda_L} + \frac{1}{\lambda_L} e^{\lambda_L \Delta_L} - \frac{1}{\lambda_L} \end{aligned} \tag{3}$$

Solving Equations (2) and (3) gives $E[t_s]$ and $E[t_L]$ as follows.

Proposition 1. *The expected lengths of green phases are given as follows.*

$$\begin{aligned} E[t_s] = \frac{(f_s - \lambda_s)(f_L - \lambda_L)}{f_L f_s - f_s \lambda_L - f_L \lambda_s} \times \left\{ \frac{\lambda_s \delta}{f_s - \lambda_s} + \frac{1}{\lambda_s} e^{\lambda_s \Delta_s} - \frac{1}{\lambda_s} \right. \\ \left. + \frac{\lambda_s}{f_s - \lambda_s} \left(\frac{\lambda_L \delta}{f_L - \lambda_L} + \frac{1}{\lambda_L} e^{\lambda_L \Delta_L} - \frac{1}{\lambda_L} \right) \right\} \end{aligned} \tag{4a}$$

and

$$E[t_L] = \frac{(f_s - \lambda_s)(f_L - \lambda_L)}{f_L f_s - f_s \lambda_L - f_L \lambda_s} \times \left\{ \frac{\lambda_L \delta}{f_L - \lambda_L} + \frac{1}{\lambda_L} e^{\lambda_L \Delta_L} - \frac{1}{\lambda_L} \right. \\ \left. + \frac{\lambda_L}{f_L - \lambda_L} \left(\frac{\lambda_s \delta}{f_s - \lambda_s} + \frac{1}{\lambda_s} e^{\lambda_s \Delta_s} - \frac{1}{\lambda_s} \right) \right\} \tag{4b}$$

From Proposition 1, it is clear that green duration in each approach is a function of the critical gaps and all other characteristics (such as discharge rate, arrival rate, and green loss) in both approaches. This demonstrates the dynamics between both approaches. As $\frac{\lambda_s}{f_s} + \frac{\lambda_L}{f_L} \to 1$, the expected green times in both approaches become infinite. This is clear by making a change to the denominator in the first term: $f_L f_s - f_s \lambda_L - f_L \lambda_s = f_L f_s (1 - \frac{\lambda_s}{f_s} - \frac{\lambda_L}{f_L})$.

The effect of the green loss δ becomes clearer when we set the critical gaps to zero. In this case, Equations (4a) and (4b) become the following.

$$E[t_s] = \frac{\lambda_s f_L \delta}{f_L f_s - f_s \lambda_L - f_L \lambda_s} \tag{5a}$$

and

$$E[t_L] = \frac{\lambda_L f_s \delta}{f_L f_s - f_s \lambda_L - f_L \lambda_s} \tag{5b}$$

We can easily have the following result.

$$\frac{E[t_s]}{E[t_L]} = \frac{\lambda_s f_L}{\lambda_L f_s}$$

Specially, the ratio of green times is proportionate to the ratio of vehicle arrival intensities if the discharge rates are equal in both approaches.

Then in this case, the expected duration of a full signal cycle becomes as follows.

$$C = E[t_s] + E[t_L] + \delta$$
$$= \frac{\delta}{1 - \frac{\lambda_s}{f_s} - \frac{\lambda_L}{f_L}} \tag{6}$$

It is clear from above that the expected cycle length is proportionate to the green loss and that it increases with the intersection saturation rate: $\frac{\lambda_s}{f_s} + \frac{\lambda_L}{f_L}$.

The following result summarizes the above discussion.

Corollary 1. *When the critical gaps are set to zero, the ratio between the expected green duration and the cycle length in both approaches is proportionate to* $\frac{\lambda_s}{f_s}$ *and* $\frac{\lambda_L}{f_L}$*, respectively. The expected cycle length is* $\frac{\delta}{1 - \frac{\lambda_s}{f_s} - \frac{\lambda_L}{f_L}}$.

The above property mimics that from the preset timing system with uniform vehicle arrivals.

VARIANCE OF GREEN TIMES

One objective of this study is to investigate the stochastic properties of the signal system cycle. For this purpose, we need to evaluate the variances of green phases. Variances of green phases are an important measure of intersection stability. They are also needed in the calculation of the average vehicle delay.

Variances in both approaches are inter-related. If the duration of the green phase in one approach becomes longer, expectedly the duration of the green phase in its conflict approach becomes longer as well. As a result, if the variance of the green duration in one approach becomes larger, expectedly the variance in its conflict approach becomes larger as well. The derivation in this section exploits this observation to establish recursive equations. Note that the following derivation implies an assumption that there exist stationary variances in both approaches.

To evaluate variance of green time, we can resort to the results from standard queueing theory when the discharge headway follows a general distribution. This will lead to results that are not transparent. Furthermore, it is still debatable whether or not this makes the model more realistic. In addition, our observation shows a very stable and constant discharge rate when vehicle queues are present. Therefore, in this section the vehicle departure headway at an intersection is seen as constant and this treatment cannot change the results far from reality.

We suppose there is a vehicle at the stop line at the beginning of the green time in the minor approach. Discharging this vehicle takes a time interval, H_s, where $H_s = \frac{1}{f_s}$. During this interval there might be several vehicles arriving to join the queue, and those vehicles also need to be discharged. This discharging procedure continues until the queue is cleared. We denote the time period for this discharging procedure by χ_1 and denote its probability distribution function by $P(x)$ where $P(x) = P\{\chi_1 \leq x\}$. If there is a queue containing i vehicles at the beginning of the green time, we denote the similar period for discharging this queue by χ_i and its related distribution by $P_i(x)$ where $P_i(x) = P\{\chi_i \leq x\}$. It is easy to see that $P_i(x)$ is an i-fold convolution of $P(x)$, which is equivalent to cleaning i queues of only one vehicle each. By the theorem of total probability, we have the relationship:

$$P\{\chi_1 \leq x\} = \sum_{i=0}^{\infty} \frac{(\lambda_s H_s)^i}{i!} e^{-\lambda_s H_s} P\{\chi_i \leq x - H_s\}$$

To prove this relationship, we only need to note that during the departure period H_s of the first vehicle, the number of arrivals follows Poisson distribution. If we let $\Gamma(s) = E(e^{-s\chi_1})$, the Laplace-Stieltjes transform of $P(x)$, then from the above analysis we have the following relationship:

$$\begin{aligned}\Gamma(s) &= \sum_{i=0}^{\infty} \frac{(\lambda_s H_s)^i}{i!} e^{-\lambda_s H_s} E\{e^{-s(\chi_i + H_s)}\} \\ &= \sum_{i=0}^{\infty} \frac{(\lambda_s H_s)^i}{i!} e^{-\lambda_s H_s} e^{-sH_s} (\Gamma(s))^i\end{aligned}$$

To get this equation we need to note that Laplace-Stieltjes transform of $P_i(x)$ is i-th product of $\Gamma(s)$. Hence we have:

$$\Gamma(s) = \exp\{H_s(\lambda_s\Gamma(s) - \lambda_s - s)\} \tag{7}$$

Using the derivatives of Equation (7) at the point where $s = 0$, we can get any moment of χ_1.

Now we can evaluate the random variable t_{sa}. Suppose in the minor approach that the red time t_L and the lost time, δ, are known. During the time interval $t_L + \delta$, the number of arrivals follows Poisson distribution. By the theorem of total probability, we have:

$$P\{t_{sa} \le x\} = \sum_{i=0}^{\infty} \frac{(\lambda_s(t_L + \delta))^i}{i!} e^{-\lambda_s(t_L+\delta)} P\{\chi_i \le x\}$$

Forming the Laplace-Stieltjes transform of $P\{t_{sa} \le x\}$, denoted by $F(s)$, we have the following equation:

$$\begin{aligned} F(s) &= \int_0^{+\infty} e^{-sx} dP\{t_{sa} \le x\} \\ &= \sum_{i=0}^{+\infty} \frac{(\lambda_s(t_L + \delta))^i}{i!} e^{-\lambda_s(t_L+\delta)} \int_0^{+\infty} e^{-sx} dP\{\chi_i \le x\} \\ &= \exp\{\lambda_s(t_L + \delta)(\Gamma(s) - 1)\} \end{aligned} \tag{8}$$

From Equation (7) it is easy to evaluate the first and second derivatives of $\Gamma(s)$ at $s = 0$:

$$\Gamma^{'}(0) = -\frac{1}{f_s - \lambda_s} \tag{9a}$$

$$\Gamma^{''}(0) = \frac{f_s}{(f_s - \lambda_s)^3} \tag{9b}$$

Hence, the expected t_{sa} conditional on $t_L + \delta$ is:

$$E[t_{sa} \mid t_L + \delta] = -F^{'}(0) = \frac{\lambda_s(t_L + \delta)}{f_s - \lambda_s} \tag{10}$$

This equation agrees with the equation in the previous section if we take expectation of $t_L + \delta$.

The variance of t_{sa} conditional on $t_L + \delta$ can be estimated as:

$$Var(t_{sa} \mid t_L + \delta) = F^{''}(0) - (F^{'}(0))^2$$
$$= \frac{\lambda_s f_s (t_L + \delta)}{(f_s - \lambda_s)^3} \tag{11}$$

To obtain unconditional variances of t_{sa} and t_{La}, we need to use the relation:

$$Var(t_{sa}) = E[Var(t_{sa} \mid t_L + \delta)] + Var(E[t_{sa} \mid t_L + \delta])$$

By using Equations (10) and (11) we have:

$$Var(t_{sa}) = \frac{\lambda_s f_s (E[t_L] + \delta)}{(f_s - \lambda_s)^3} + \frac{\lambda_s^2}{(f_s - \lambda_s)^2} Var(t_L) \tag{12}$$

In previous analyses, the green time contains two periods: queue clearance time and free flow time. Because of the Markov property, these two periods are independent of each other. As a result, we have:

$$Var(t_s) = Var(t_{sa}) + Var(t_{sb})$$

According to Wang (2007), $Var(t_{sb}) = -\frac{1}{\lambda_s^2} - \frac{2\Delta_s e^{\lambda_s \Delta_s}}{\lambda_s} + \frac{e^{2\lambda_s \Delta_s}}{\lambda_s^2}$. Therefore, we have the equations for variances:

$$Var(t_s) = \frac{\lambda_s f_s (E[t_L] + \delta)}{(f_s - \lambda_s)^3} + \frac{\lambda_s^2}{(f_s - \lambda_s)^2} Var(t_L) - \frac{1}{\lambda_s^2} - \frac{2\Delta_s e^{\lambda_s \Delta_s}}{\lambda_s} + \frac{e^{2\lambda_s \Delta_s}}{\lambda_s^2} \tag{13a}$$

and similarly,

$$Var(t_L) = \frac{\lambda_L f_L (E[t_s] + \delta)}{(f_L - \lambda_L)^3} + \frac{\lambda_L^2}{(f_L - \lambda_L)^2} Var(t_s) - \frac{1}{\lambda_L^2} - \frac{2\Delta_L e^{\lambda_L \Delta_L}}{\lambda_L} + \frac{e^{2\lambda_L \Delta_L}}{\lambda_L^2} \tag{13b}$$

It is clear now that $Var(t_s)$ and $Var(t_L)$ are linear functions of each other. Solving Equations (13a) and (13b) gives values of $Var(t_s)$ and $Var(t_L)$ as follows.

Proposition 2. *The equilibrium solutions for variances of green phases are given as follows.*

$$
\begin{aligned}
Var(t_s) = {} & \left(1-\frac{\lambda_s{}^2\lambda_L^2}{(f_s-\lambda_s)^2(f_L-\lambda_L)^2}\right)^{-1}\left(\frac{\lambda_s f_s(E[t_L]+\delta)}{(f_s-\lambda_s)^3}+\frac{\lambda_s^2}{(f_s-\lambda_s)^2}\right. \\
& \left.\times\left[\frac{\lambda_L f_L(E[t_s]+\delta)}{(f_L-\lambda_L)^3}+Var(t_{Lb})\right]+Var(t_{sb})\right)
\end{aligned}
\tag{14a}
$$

$$
\begin{aligned}
Var(t_L) = {} & \left(1-\frac{\lambda_s{}^2\lambda_L^2}{(f_s-\lambda_s)^2(f_L-\lambda_L)^2}\right)^{-1}\left(\frac{\lambda_L f_L(E[t_s]+\delta)}{(f_L-\lambda_L)^3}+\frac{\lambda_L^2}{(f_L-\lambda_L)^2}\right. \\
& \left.\times\left[\frac{\lambda_s f_s(E[t_L]+\delta)}{(f_s-\lambda_s)^3}+Var(t_{sb})\right]+Var(t_{Lb})\right)
\end{aligned}
\tag{14b}
$$

where $Var(t_{sb})=-\frac{1}{\lambda_s^2}-\frac{2\Delta_s e^{\lambda_s\Delta_s}}{\lambda_s}+\frac{e^{2\lambda_s\Delta_s}}{\lambda_s^2}$ and $Var(t_{Lb})=-\frac{1}{\lambda_L^2}-\frac{2\Delta_L e^{\lambda_L\Delta_L}}{\lambda_L}+\frac{e^{2\lambda_L\Delta_L}}{\lambda_L^2}$. $E[t_s]$ and $E[t_L]$ are given by Equations (4a) and (4b).

It is interesting to evaluate the green phases when the critical gap is set to zero, as indicated in the optimal solution later. This strategy is in fact the queue control strategy studied by Kruger *et al*. (1990). In this case, both $Var(t_{sa})$ and $Var(t_{La})$ become zero. Equations (14a) and (14b) become:

$$
Var(t_s)=C\left(\frac{(f_L-\lambda_L)^2\lambda_s f_s(E[t_L]+\delta)}{(f_s-\lambda_s)}+\frac{\lambda_s^2\lambda_L f_L(E[t_s]+\delta)}{(f_L-\lambda_L)}\right)
\tag{15a}
$$

and

$$
Var(t_L)=C\left(\frac{(f_s-\lambda_s)^2\lambda_L f_L(E[t_s]+\delta)}{(f_L-\lambda_L)}+\frac{\lambda_L^2\lambda_s f_s(E[t_L]+\delta)}{(f_s-\lambda_s)}\right)
\tag{15b}
$$

where

$$C = \frac{1}{\left(1 - \frac{\lambda_s}{f_s} - \frac{\lambda_L}{f_L}\right)\left(f_s^2 f_L^2 \left(1 - \frac{\lambda_s}{f_s} - \frac{\lambda_L}{f_L}\right) + 2\lambda_s f_s \lambda_L f_L\right)} \tag{16}$$

With Equations (15a) and (15b), we can examine the impact of saturation rate on variances and intersection stability. Now we take a look at the constant C, represented as Equation (16) at the right-hand sides of both Equations (15a) and (15b).

It is easily seen by Equations (5a) and (5b) that as $Var(t_L$, the expected green times increase at a rate of $O\left(\left[1 - \frac{\lambda_s}{f_s} - \frac{\lambda_L}{f_L}\right]^{-1}\right)$. But the variances shown by Equations (15a) and (15b) approach infinity much faster, at a rate of $O\left(\left[1 - \frac{\lambda_s}{f_s} - \frac{\lambda_L}{f_L}\right]^{-1}\right)$. This means that as the traffic flow reaches the capacity of an intersection, the system quickly becomes instable. Combined with Equations (5a) and (5b), Equations (15a) and (15b) imply the following result.

Proposition 3. *Given an actuated intersection with zero critical gap, both the expected values and variances of the green times increase in a linear manner with the total green time loss* δ.

The above result indicates the critical role of green time loss in the performance of an intersection.

SYMMETRIC INTERSECTIONS

In particular, if the two approaches are symmetric, we have $\lambda_s = \lambda_L = \lambda$ and $f_s = f_L = f$. The following result becomes obvious when the critical gaps are further set to zero.

$$E[t_s] = E[t_L] = \frac{\lambda\delta}{f - 2\lambda} \tag{17}$$

In this case, the entire cycle length becomes $\frac{\lambda\delta}{f - 2\lambda}$, a linear function of the green time loss δ .

Furthermore, the variances in this case increase proportionately with δ , as shown below.

$$Var(t_s) = Var(t_L) = \frac{\lambda\delta}{(f - 2\lambda)^2} \tag{18}$$

It is interesting to note that $\frac{\sqrt{Var(t_s)}}{E[t_s]} = \frac{\sqrt{Var(t_L)}}{E[t_L]} = \frac{1}{\sqrt{\lambda\delta}}$, which is a constant. This result agrees with the conclusion by Newell (1969), which indicates the green time distribution can be approximately Gamma type.

CHAPTER 3:
VEHICLE DELAY

The cycles of green and red times vary in length due to the random nature of traffic arrivals. Considering that each signal cycle repeats the same process with an identical probability distribution, we may treat the cycle times as a renewal process. The renewal process must have a stationary state in which a certain probability distribution governs the cycle time including both the green and red. We will therefore develop analytical results in this chapter based on this stationary property.

VEHICLE DELAY FOR A SINGLE CYCLE

We start with the minor approach, assuming that t_L and δ are given. The waiting time has two components. When the signal turns green in the minor approach, there has been a queue, the number of vehicles of which is denoted by N_s; there is also a corresponding expected waiting time that has been incurred, denoted by W_s^1. The other component of the waiting time is determined during the period in which the queue is being discharged. We denote this waiting time by W_{sa}. Be aware that there is no new queue during the free flow period t_{sb} and t_{Lb}, as explained before. Note that N_L, W_L^1, and W_{La} are the corresponding notations for the major approach. By approximating the total waiting time based on continuous methods, we have the following result.

Proposition 4. *At the time when the green light first turns on in the minor approach, the total vehicle delay, denoted by W_s^1, of those arrivals N_s, during the red signal period is given as follows.*

$$E[W_s^1] = \frac{1}{2}\lambda_s(Var(t_L) + E^2[t_L] + 2\delta E[t_L] + \delta^2)$$

and, similarly for the major approach,

$$E[W_L^1] = \frac{1}{2}\lambda_L(Var(t_s) + E^2[t_s] + 2\delta E[t_s] + \delta^2)$$

Proof.

According to Ross (1997), with a given number of arrivals from a Poisson process during a time period, the distribution of each arrival is uniform within the given time interval. Therefore, the expected waiting time of the queue can be calculated by conditioning on the length of queue:

$$\begin{aligned}
E[W_s^1] &= E[E[E[E[W_s^1 \mid X(t_L+\delta)]]] \mid t_L]] \\
&= E[(t_L+\delta)\times\lambda_s\times\frac{1}{2}\times(t_L+\delta)] \\
&= \frac{1}{2}\times\lambda_s\times E[(t_L+\delta)^2] \\
&= \frac{1}{2}\times\lambda_s\times(Var(t_L)+E^2(t_L)+2\delta E[t_L]+\delta^2)
\end{aligned}$$

Similarly, we can show the result for $E[W_L^1]$.

□

We further have the following results for W_{sa} and W_{La}, respectively.

Proposition 5. *The waiting times during queue discharge are given as follows.*

$$E[W_{sa}] = \frac{1}{2}(f_s-\lambda_s)(Var(t_{sa})+E^2[t_{sa}]) \tag{19a}$$

And, similarly,

$$E[W_{La}] = \frac{1}{2}(f_L-\lambda_L)(Var(t_{La})+E^2[t_{La}]) \tag{19b}$$

Proof.

We only prove the result in the minor approach. The result for the major approach can be obtained similarly.

We condition on queue discharge time t_{sa}.

$$
\begin{aligned}
E[W_{sa} \left| t_{sa} \right.] &= E\left[\int_0^{t_{sa}} [X(t_L + \delta) + X(t) - f_s t] dt \left| t_{sa} \right. \right] \\
&= E\left[\int_0^{t_{sa}} X(t_L + \delta) dt \left| t_{sa} \right. \right] + E\left[\int_0^{t_{sa}} (\lambda_s t - f_s t) dt \left| t_{sa} \right. \right] \\
&= (f_s - \lambda_s) t_{sa}^2 + \frac{1}{2}(\lambda_s - f_s) t_{sa}^2 \\
&= \frac{1}{2}(f_s - \lambda_s) t_{sa}^2
\end{aligned}
$$

The third equality above uses the fact that $E[X(t_L + \delta) \mid t_{sa}] = (f_s - \lambda_s) t_{sa}$, according to Equation (1).

We therefore have:

$$
\begin{aligned}
E[W_{sa}] &= E[E[W_{sa} \mid t_{sa}]] \\
&= \frac{1}{2}(f_s - \lambda_s)(Var(t_{sa}) + E^2[t_{sa}])
\end{aligned}
$$

To summarize, the total expected vehicle delay during the period of a signal cycle at the intersection can be expressed in a closed form by $E[W_s^1] + E[W_L^1] + E[W_{sa}] + E[W_{La}]$ explicitly. We can calculate the expected waiting time by substituting the equations for the expected values and variances.

VEHICLE DELAY PER UNIT TIME

The signal cycling can be considered as a renewal process, from the beginning of green time to the beginning of green time again. According to the renewal theory, the average vehicle delay per unit time over this renewal process is the ratio between the total expected vehicle delay in a cycle and the expected cycle length. This average vehicle delay can therefore be expressed in terms of Δ_s and Δ_L with given parameters λ_s, λ_L, f_s, and f_L. We denote by $F(\Delta_L, \Delta_s)$ the function of average vehicle delay per unit time. The function $F(\Delta_L, \Delta_s)$ can be expressed in the following way.

$$
F(\Delta_L, \Delta_s) = \frac{E[W_s^1] + E[W_L^1] + E[W_{sa}] + E[W_{La}]}{E[t_s] + E[t_L] + \delta} \tag{20}
$$

The closed form (20) of $F(\Delta_L, \Delta_s)$ consists of a large number of terms, making it very inconvenient to analytically examine its properties in terms of concavity or convexity. However, our observation is that the delay function appears to be a coercive function in the critical gap. This means that as the critical gap increases, the delay increases and tends to infinity. In addition, it does not appear appropriate to obtain the minimizer by identifying the stationary point whose first order derivative is zero, as in Darroch *et al*. (1964). However, we can easily study this function numerically by modern computing techniques.

Figure 3 in Appendix B graphically demonstrates an example delay function when $\delta = 2.0$, $\lambda_s = 0.15$, $\lambda_L = 0.25$, $f_s = 0.6$, and $f_L = 0.6$. We have conducted a number of numerical studies with combinations of different arrival rates, discharging rates, and green loss. Surprisingly, we find that $F(\Delta_L, \Delta_s)$ is minimized when Δ_s and Δ_L are both set to zero in an overwhelming majority of cases. This result defies a popular practice that configures positive critical gaps. However, the observation of zero critical gap in those cases makes sense. Expectedly, it takes $E[t_s]$ or $E[t_L]$ to clear the queue. After the queue has been cleared, it is very likely that a few vehicles will be waiting in the conflict approach. At this time, it is more desirable to switch the green signal. A negative impact of implementing a positive critical gap in the major approach is that the minor approach will have a long queue and a long time to clear it; therefore, there will be a long queue accordingly in the major approach in the following cycle. In one word, extension of green time in one approach expectedly increases vehicle waiting time in ALL the approaches in the immediate next cycle.

Nevertheless, when the minor approach has an extremely low arrival rate, the optimal critical gap in the major approach could be quite significant, as in Figure 4 in Appendix B. In Figure 4, the discharge rates in both approaches are 0.6 vehicles per second. The arrival rates are 0.02 and 0.25 vehicles per second, respectively. The green loss during a cycle is 4 seconds. The average vehicle delay reaches its minimum when $\Delta_s = 0.0$ and $\Delta_L = 4.4$ seconds. Of course, this extreme case can be better addressed by a different control scheme. Now we are able to evaluate intersection performance in various settings. Table 1 and 2 are sample test results where $f_L = f_s = 0.6$ vehicles per second.

Table 1: Optimal critical gap and intersection performance with $\delta = 2.0$

λ_s / λ_L	Δ_s / Δ_L	$Var(t_s)$	$Var(t_L)$	$F(\cdot)$
0.02/0.25	0.0/4.4	0.8	31.5	0.398
0.05/0.25	0.0/2.8	1.6	14.1	0.702
0.08/0.25	0.0/1.8	2.5	12.5	0.989
0.15/0.25	0.0/0.0	6.2	15.4	1.775
0.20/0.25	0.0/0.0	16.1	24.6	2.616

Table 2: Optimal critical gap and intersection performance with $\delta = 4.0$

λ_s / λ_L	Δ_s / Δ_L	$Var(t_s)$	$Var(t_L)$	$F(\cdot)$
0.02/0.25	0.0/5.6	1.3	82.5	0.642
0.05/0.25	0.0/4.4	3.1	39.1	1.022
0.08/0.25	0.0/2.4	4.5	24.6	1.499
0.15/0.25	0.0/0.0	12.3	30.8	2.550
0.20/0.25	0.0/0.0	32.3	49.1	3.733

From Table 1 and 2, we have the following observations.

Observation 1. *The optimal critical gap always appears zero in the minor approach.*

Regarding the test examples, our purpose is to examine a special control scheme, not the optimal scheme, under different situations. Some of the examples to which this special scheme is applied might not appear as a good fit. Those examples may justify a better scheme. For instance, the first example in Table 1 may warrant a signal in which traffic from the minor direction always yields to that in the major direction.

In fact, the result of setting both Δ_s and Δ_L be zero is the same with queue control strategy, whose implementation was studied by Kruger *et al*. (1990). Although the optimal solution is setting the critical gaps to zero in the model, in real application, the critical gap can be set to a small interval in order to detect the time of clearance of the queue.

ENVIRONMENTAL CONCERNS

We compare an optimal scheme of minimizing vehicle environmental impact (e.g. delay plus the number of stop-and-go) with one that only minimizes vehicle delay, as in the previous section. We use P_s and P_L for the total cost including environmental impact for the minor and major roads, respectively. P_s and P_L each has two components: cost associated with the number of stop-and-goes, and cost associated with the emissions due to waiting in queues. We use n_s and n_L for the expected number of stop-and-goes in the two directions respectively. We use α to convert the stop-and-goes into an equivalent measure of waiting time.

The number of stops can be easily derived with a given queue clearance time.

$$\begin{aligned} n_s &= f_s E[t_{sa}] \\ n_L &= f_L E[t_{La}] \end{aligned} \tag{21}$$

We naturally have the following total cost expressions.

$$\begin{aligned} P_s &= \alpha n_s + W_s^1 + W_{sa} \\ P_L &= \alpha n_L + W_L^1 + W_{La} \end{aligned} \tag{22}$$

In light of previous analyses, the new unit time cost with the environmental consideration is expressed as follows.

$$F^*(\Delta_L, \Delta_s) = \frac{P_s + P_L}{E[t_s] + E[t_L] + \delta} \tag{23}$$

The optimal control considering environmental issues is to minimize the above expression. Comparing this expression and Equation (20), the difference is a separated term that has nothing to do with the variance of green time. From this respect, the environmental concerns are traded off with the concerns of delay.

To compare this objective function with overall delay, we calculate the optimal solution for each case in Table 1 and 2. See Table 3 and 4 for the results where $\alpha = 1$ and $f_L = f_s = 0.6$. From Table 3 and 4, we find the same conclusion that $F^*(\Delta_L, \Delta_s)$ is minimized when Δ_s and Δ_L are both set to zero in the case where arrival rates in the two directions are comparable.

Table 3: Optimal critical gap and environmental concerns with $\delta = 2.0$

λ_s / λ_L	Δ_s / Δ_L	$Var(t_s)$	$Var(t_L)$	$F^*(\cdot)$
0.02/0.25	0.0/5.0	0.9	48.4	0.497
0.05/0.25	0.0/3.2	1.7	16.9	0.879
0.08/0.25	0.0/2.4	2.9	14.7	1.232
0.15/0.25	0.0/0.0	6.2	15.4	2.175
0.20/0.25	0.0/0.0	16.1	24.6	3.067

Table 4: Optimal critical gap and environmental concerns with $\delta = 4.0$

λ_s / λ_L	Δ_s / Δ_L	$Var(t_s)$	$Var(t_L)$	$F^*(\cdot)$
0.02/0.25	0.0/6.4	1.5	122.9	0.756
0.05/0.25	0.0/4.0	2.9	35.4	1.289
0.08/0.25	0.0/2.6	4.6	25.6	1.765
0.15/0.25	0.0/0.0	12.3	30.8	2.950
0.20/0.25	0.0/0.0	32.3	49.1	4.183

OPTIMAL CONTROL TO ACCOMMODATE COMMERCIAL VEHICLE MOVEMENT

We have discussed the optimal control strategy for homogeneous traffic. However, this is not enough for signal operations within urban areas, because the impact of freight traffic and mobility of commercial vehicles need to be considered as well. Usually, commercial vehicles lead to larger delay than passenger cars do. Here, we consider a traffic mix of both passenger cars and commercial vehicles simultaneously. We assume that a certain percentage, β, of vehicles are commercial vehicles and delay of a commercial vehicle has a cost γ times that of a passenger car for the same unit of waiting time during queue dissipation. In addition, we assume the percentage of commercial vehicles does not change the expectation and variance of green time. This assumption is strong, but we believe it serves our purpose of analysis to a certain degree. We will examine in the following how the previously derived results are altered due to the change of traffic mix. For such heterogeneous traffic, $E[W_{sa}] + E[W_{La}]$ change to $((1-\beta) + \gamma\beta)(E[W_{sa}] + E[W_{La}])$. Accordingly, the new unit time cost with heterogeneous traffic consideration is expressed as follows:

$$F^{**}(\Delta_L, \Delta_s) = \frac{E[W_s^1] + E[W_L^1] + ((1-\beta) + \gamma\beta)(E[W_{sa}] + E[W_{La}])}{E[t_L] + E[t_s] + \delta} \quad (24)$$

The optimal control considering heterogeneous traffic is to minimize Equation (24). For comparison to each case in Table 1 and 2, we calculate the optimal solution when $\beta = 0.05$ and $\gamma = 1.2$. From the results shown in Table 5 and 6, we see a similar conclusion that Equation (24) is minimized when Δ_s and Δ_L are both set to zero in the case where arrival rates in the two directions are comparable. Presence of commercial vehicles slightly increases the green extension in the according approach.

Table 5: Optimal critical gap and heterogeneous traffic with $\delta = 2.0$

λ_s / λ_L	Δ_s / Δ_L	$Var(t_s)$	$Var(t_L)$	$F^{**}(\cdot)$
0.02/0.25	0.0/4.4	0.8	31.5	0.400
0.05/0.25	0.0/2.8	1.6	14.1	0.705
0.08/0.25	0.0/1.8	2.5	12.5	0.994
0.15/0.25	0.0/0.0	6.2	15.4	1.784
0.20/0.25	0.0/0.0	16.1	24.6	2.630

Table 6: Optimal critical gap and heterogeneous traffic with $\delta = 4.0$

λ_s / λ_L	Δ_s / Δ_L	$Var(t_s)$	$Var(t_L)$	$F^{**}(\cdot)$
0.02/0.25	0.0/5.6	1.3	82.5	0.644
0.05/0.25	0.0/3.4	2.6	27.4	1.096
0.08/0.25	0.0/2.2	4.3	23.7	1.503
0.15/0.25	0.0/0.0	12.3	30.8	2.562
0.20/0.25	0.0/0.0	32.3	49.1	3.751

CHAPTER 4: GENERAL VEHICLE HEADWAY UNDER HEAVY TRAFFIC

Regarding modeling of intersection control, concerns rest on how a strategy performs with general arrival headway. In this case, we assume that vehicle arrivals follow a general renewal process and that vehicle headways are independent and identically distributed (i.i.d.) random variables whose density function is denoted by $f(\cdot)$. The arrival intensity is λ, and we use H for the stochastic headway. Clearly, we have $\lambda E[H]=1$. We use σ^2 for the variance of H, and the subscripts L and s denote the major and minor direction, respectively. In the following, notations without subscriptions apply to both directions.

The following result is critical to deriving the results in the general case.

Theorem 1. *Given a constant* t*, the following assumptions approximately hold for both directions under heavy traffic:*

$$E[X(t)] \approx \lambda t \tag{25a}$$

$$Var(X(t)) \approx kt \tag{25b}$$

where $k \approx \dfrac{\sigma^2}{E^3[H]}$.

In the case of Poisson arrival (e.g., headway with exponential distribution), we can easily verify that $k \approx \lambda$, in which case the values in Theorem 1 are accurate. In general cases under heavy traffic where vehicle headway is small compared to the green time in each direction, Theorem 1 is a good approximation. Its proof is seen in the Appendix A.

With linear assumption about $X(t)$, we can easily derive the results in the general case corresponding to propositions in the previous sections.

Another important result is about waiting time corresponding to Proposition 4. We have the following.

Proposition 6. *Under heavy traffic, the expected vehicle waiting time* $w(t)$ *is approximately* $\frac{\lambda t^2}{2}$,

i.e., $w(t) \approx \frac{\lambda t^2}{2}$, where t is the time that the signal has been red in the direction of interest.

Proof.

We have:

$$w(t) = \int (t - l + w(t-l)) f(l) dl$$

We only need to check if the assumption can approximately satisfy the above equation.

$$\begin{aligned}
\text{left-handside} &= \frac{1}{2}\lambda t^2 \\
\text{right-handside} &= t - E[H] + \int_0^t \frac{1}{2}\lambda (t-l)^2 f(l) dl \\
&= t - E[H] + \frac{1}{2}\lambda (t^2 - 2tE[H] + \sigma^2 + E^2[H]) \\
&\approx \frac{1}{2}\lambda t^2 + \frac{1}{2}\lambda\sigma^2 - \frac{1}{2}E[H]
\end{aligned}$$

We call the remnant in the right-hand side $\frac{1}{2}\lambda\sigma^2 - \frac{1}{2}E[H]$ error term. The assumption can make both sides of the equation approximately equal, considering t outweighs $E[H]$ and t^2 outweighs σ^2. In the following, we show how t^2 outweighs σ^2.

$$\begin{aligned}
\frac{1}{2}\lambda\sigma^2 &= \frac{1}{2}\lambda v E[H]\sigma \\
&= \frac{1}{2}\lambda\sigma \\
&= \frac{1}{2}v^2 E[H]
\end{aligned}$$

where v is the coefficient of variation (CV) between the standard deviation σ and the expected headway $E[H]$. If CV is a finite value, then the error compared with the major term

$\frac{1}{2}\lambda t^2$ tends to zero in heavy traffic. In Poisson arrivals as a special case, the error term is zero, which can be verified by interested readers.

Next, we characterize the green extension process in each direction with general vehicle headway. The notation has the same meaning as in the previous sections. Derivation of the results is seen in Wang *et al*. (2009).

$$E[t_{sb}] = \frac{\int_0^{\Delta_s} tf(t)dt}{1 - F(\Delta_s)} \tag{26a}$$

$$E[t_{Lb}] = \frac{\int_0^{\Delta_L} tf(t)dt}{1 - F(\Delta_L)} \tag{26b}$$

$$Var(t_{sb}) = \frac{\int_0^{\Delta_s} t^2 f(t)dt}{1 - F(\Delta_s)} + E^2[t_{sb}] \tag{26c}$$

$$Var(t_{Lb}) = \frac{\int_0^{\Delta_L} t^2 f(t)dt}{1 - F(\Delta_L)} + E^2[t_{Lb}] \tag{26d}$$

Following the same process as in the previous section and using the new results above, we have the following conclusion accordingly.

Proposition 7. *The expected lengths of green phases are given as follows.*

$$\begin{aligned} E[t_s] = & \frac{(f_s - \lambda_s)(f_L - \lambda_L)}{f_L f_s - f_L \lambda_s - f_s \lambda_L} \times \left\{ \frac{\lambda_s \delta}{f_s - \lambda_s} + \frac{\int_0^{\Delta_s} tf(t)dt}{1 - F(\Delta_s)} \right. \\ & \left. + \frac{\lambda_s}{f_s - \lambda_s} \left(\frac{\lambda_L \delta}{f_L - \lambda_L} + \frac{\int_0^{\Delta_L} tf(t)dt}{1 - F(\Delta_L)} \right) \right. \end{aligned} \tag{27a}$$

$$E[t_L]=\frac{(f_s-\lambda_s)(f_L-\lambda_L)}{f_Lf_s-f_L\lambda_s-f_s\lambda_L}\times\left\{\frac{\lambda_L\delta}{f_L-\lambda_L}+\frac{\int_0^{\Delta_L}tf(t)dt}{1-F(\Delta_L)}\right.$$
$$\left.+\frac{\lambda_L}{f_L-\lambda_L}\left(\frac{\lambda_s\delta}{f_s-\lambda_s}+\frac{\int_0^{\Delta_s}tf(t)dt}{1-F(\Delta_s)}\right)\right\} \tag{27b}$$

To assess the variance of green time, we adopt the cumulative queuing curves, as illustrated in Figure 5 of Appendix B. The x-axis represents time from the beginning of red time, and the y-axis represents the length of cumulative queue. In Figure 5, the solid curve represents expected cumulative queue length, which becomes zero at the end of green time $E[t_{sa}]$; two dashed lines represent two extreme realizations due to the stochastic arrival and departure. For convenient analysis, we assume that the lower curve is extended to the point B with x-coordinate $E[t_{sa}]$. Thus, at point $E[t_{sa}]$, the fluctuation of queue length is between point A and point B. Since the expected queue length is zero, given red time $t_L+\delta$, such amplitude of queue fluctuation can be treated as standard deviation of total arrivals and departures during one cycle. On the other hand, we can use the standard deviation of green time to approximate the amplitude from the time point $E[t_{sa}]$ to the point where the cumulative queue curve and x-axis intersect. Under heavy traffic, the slope of any cumulative curve for queues near the time point $E[t_{sa}\,|\,t_L+\delta]$ can be approximated by $f_s-\lambda_s$. From the geometric illustration in Figure 5, we can use $f_s-\lambda_s$ to approximate the ratio of two standard deviations of (1) total arrivals and departures during one cycle and (2) green time. Similar discussion has been provided in literature, for example, Newell (1965). Given $t_L+\delta$, since departure rate is constant, the variance of departures is negligible during green time $E[t_{sa}\,|\,t_L+\delta]$. Therefore, we have the following approximate relation.

$$(f_s-\lambda_s)^2Var(t_{sa}\,|\,t_L+\delta)=Var(X(t_L+\delta+E[t_{sa}\,|\,t_L+\delta])) \tag{28}$$

Accordingly,

$$Var(t_{sa}\,|\,t_L+\delta)=\frac{k(t_L+\delta+E[t_{sa}\,|\,t_L+\delta])}{(f_s-\lambda_s)^2}=\frac{kf_s(t_L+\delta)}{(f_s-\lambda_s)^3} \tag{29}$$

When the arrivals follow Poisson distribution, the above equation changes to be Equation (11). By using $Var(t_{sa}) = E[Var(t_{sa} \mid t_L + \delta)] + Var(E[t_{sa} \mid t_L + \delta])$ and $Var(t_s) = Var(t_{sa}) + Var(t_{sb})$, the variance of green time is easy to derive.

Proposition 8. *The variances of green phases are given as follows.*

$$Var(t_s) = \frac{kf_s(t_L + \delta)}{(f_s - \lambda_s)^3} + \frac{\lambda_s^2 Var(t_L)}{(f_s - \lambda_s)^2} + \frac{\int_0^{\Delta_s} t^2 f(t)dt}{1 - F(\Delta_s)} + E^2[t_{sb}] \tag{30a}$$

and

$$Var(t_L) = \frac{kf_L(t_s + \delta)}{(f_L - \lambda_L)^3} + \frac{\lambda_L^2 Var(t_s)}{(f_L - \lambda_L)^2} + \frac{\int_0^{\Delta_L} t^2 f(t)dt}{1 - F(\Delta_L)} + E^2[t_{Lb}] \tag{30b}$$

Regarding waiting time, we can easily find that Propositions 4 and 5 remain valid given Theorem 1. The only change to the calculation in the Poisson process is to substitute for the new expected values and variances in derivation.

CHAPTER 5:
DISCUSSION AND CONCLUSION

This research studies a vehicle-actuated signal control scheme in which the critical gap is the only control variable. For an isolated intersection, it concludes that the critical gap in most cases adds to vehicle delay. The finding in this research, when translated to practice, essentially says that the green time ensures only a saturation discharge rate for queue clearance, which implies the queue control strategy. Technical means to implement such a scheme are not discussed in this paper. Readers might refer to Kruger *et al*. (1990) for a discussion in this regard.

The finding in this paper is insightful to intersections of three or more approaches. In the case of an intersection between two one-ways, extension of green time in approach A causes delay in approach B and additional delay in approach A from the next cycle and on, which is the potential cost of green extension. In contrast, the benefit of it is possibly having some vehicles passing through the intersection without stops in the current extension period. Obviously, in the case of an intersection with two one-ways, the cost outweighs the benefit. In the case of an intersection between three or more approaches, the potential cost of resulting delay would be even perceivably larger. We can be reasonably confident that the principle of clearing only queueing vehicles and allowing only for saturation flow rate at the intersection during the green time holds for general cases.

It is interesting to find some similarities between the actuated signal system and the pre-timed one. The latter usually assumes a uniform vehicle arrival. It has the same green time allocation as the expected green time in an actuated signal system when the critical gap is set to zero. In both cases, the green phases are designed to discharge vehicular queues only. The difference is that the actuated signal system readily adapts to random arrivals.

It is worth mentioning that we study a special control scheme in which the critical gap is the *only* control variable. The findings here could change if the number of queueing vehicles in the conflict approach is also considered as a condition for phase switching. Finally, we believe that the insight into the impact of prolonged green time in one approach onto the entire performance of the intersection applies to general intersections in some way, and therefore has its general practical and theoretical significance.

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APPENDIX

A. DERIVATION

In this section, we provide proof for the linear results in Theorem 1. We use $X(t)$ for the random number of arrivals in a direction during time t. We do not specify directions in this justification. Derivation applies to both directions.

The number of arrivals in a direction follows the following relationship, assuming a known t.

$$\begin{aligned} X(t) &= \int_0^t (X(t-l)+1) f(l) dt \\ &= \int_0^t X(t-l) f(l) dt + F(t) \end{aligned} \quad (*)$$

where $f(\,)$ represents the probability density function of the vehicle headway. Equation (*) conditions on the first vehicle arrival. When the first vehicle arrives at time l at a probability $f(l)dt$, the total arrival (including the current one and potentials) is $X(t-l)+1$.

Taking expectation on both sides of Equation (*), we have the following:

$$E[X(t)] = \int_0^t E[X(t-l)] f(l) dt + F(t)$$

Substitute $E[X(t)] = \lambda t$,

$$\begin{aligned} \text{left-handside} &= \lambda t \\ \text{right-handside} &= \int_0^t \lambda(t-l) f(l) dl + F(t) \\ &= \lambda t F(t) - \lambda \int_0^t l f(l) dl + F(t) \\ \text{(when large t)} &\approx \lambda t - \lambda E[H] + F(t) \\ &= \lambda t \end{aligned}$$

When t is large enough compared with the vehicle headway, $F(t) \to 1.0$. In this derivation, we directly use $F(t) = 1.0$. The right-hand side approximately equals the left-hand side when t is large compared to vehicle headway, as in $\lambda E[H] = 1$. Therefore, it is reasonable to assume a linear relationship between the expected number of arrivals and the length of time during heavy traffic. The assumption $E[X(t)] = \lambda t$ is often directly used in transportation research. We conjecture that when two or more vehicles are expectedly queued at the start of green time in a direction, this assumption should work well.

We continue to prove the linear assumption about variance. The variance of X satisfies the following relationship that is conditional on the first vehicle arrival time l.

$$V(X(t)) = V(E[X(t)] \mid l) + E[V(X(t) \mid l)] \tag{**}$$

Note that if the first vehicle arrival takes place at time l, the total expected number of arrivals in t is $E(X(t-l))+1$. If we continue to use assumption $x(t) = \lambda t$, and if the first vehicle arrives at time $l \le t$, whose probability is $f(l)dl$, the total conditional expected arrival is $\lambda(t-l)+1$. Then we have:

$$\begin{aligned} V(E[X(t)] \mid l) &= \int_0^t [\lambda(t-l)+1-\lambda t]^2 f(l)dl \\ &= F(t) - 2\lambda \int_0^t l f(l)dl + \lambda^2 \int_0^t l^2 f(l)dl \\ &= F(t) - 2\lambda E[H] + \lambda^2 (E^2[H] + \sigma^2) \end{aligned}$$

where σ^2 represents the variance of vehicle headway.

Furthermore, we have:

$$\begin{aligned} E[V(X(t)) \mid l] &= \int_0^t V(X(t-l)) f(l)dl \\ &\approx \int_0^t k(t-l) f(l)dl \\ &= kt\int_0^t f(l)dl - k\int_0^t tf(l)dl \\ &\approx kt - kE[H] \end{aligned}$$

We will see how the linear assumption holds in Equation (**).

$$\begin{aligned} \text{left-handside} &= \mathrm{k}t \\ \text{right-handside} &= F(t) - 2\lambda E[H] + \lambda^2 (E^2[H] + \sigma^2) + kt - kE[H] \\ &= kt + 1 - 2 + \lambda^2 E^2[H] + \lambda^2 \sigma^2 - kE[H] \\ &= kt + \frac{\sigma^2}{E^2[H]} - kE[H] \end{aligned}$$

In order to have both sides equal, we would have to have:

$$k = \frac{\sigma^2}{E^3[H]}$$

As a special case, we have $k = \lambda$ when arrivals follow a Poisson process, in which case the assumption is accurate and true. Note that the assumption about the mean number of vehicle arrivals holds at heavy traffic and is close to the true value in most practical approximation in practice. In particular, the variance is accurate in heavy traffic as the assumption about mean is accurate.

B. FIGURES

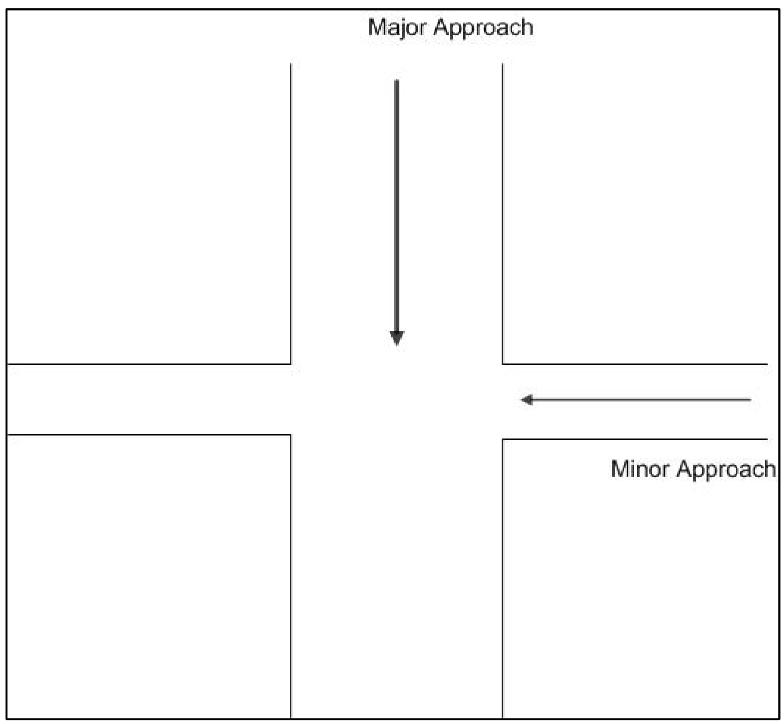

Figure 1: A major-minor intersection

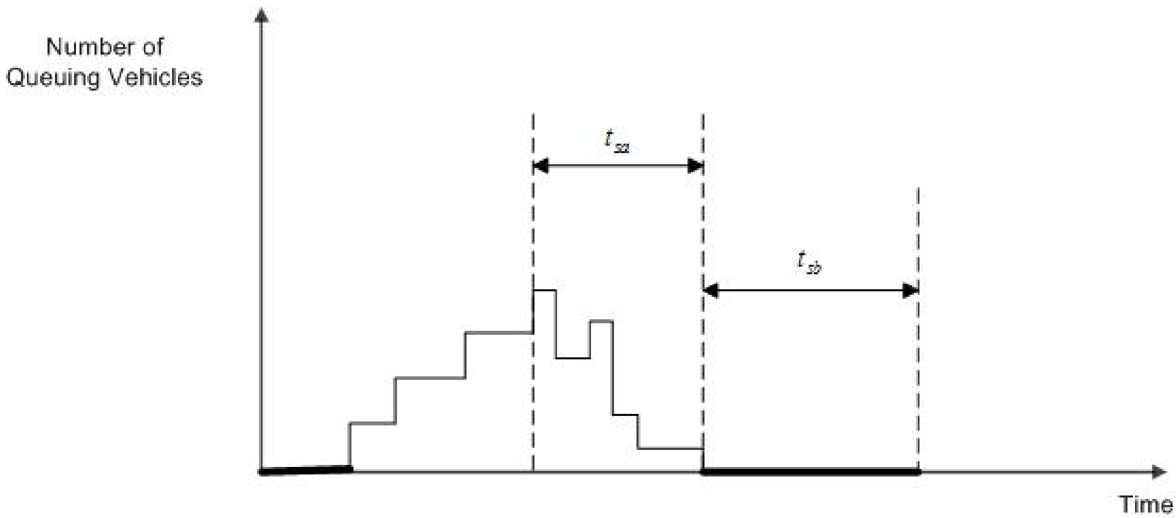

Figure 2: An illustrative queueing process in minor approach

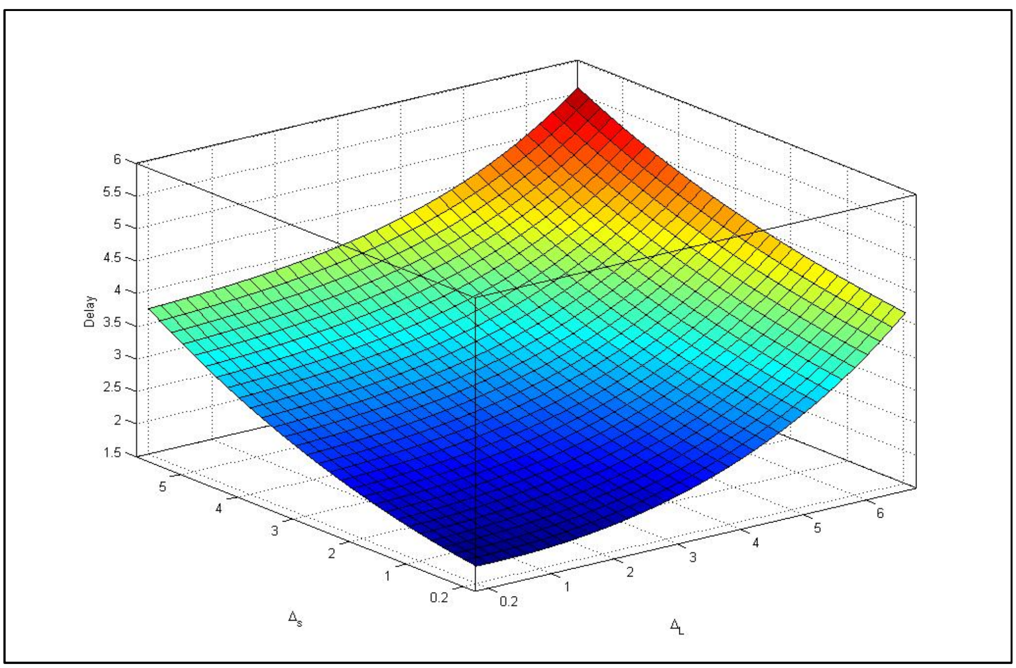

Figure 3: Example average delay per unit time with $\delta = 2.0$ **,** $\lambda_s = 0.15$ **and** $\lambda_L = 0.25$

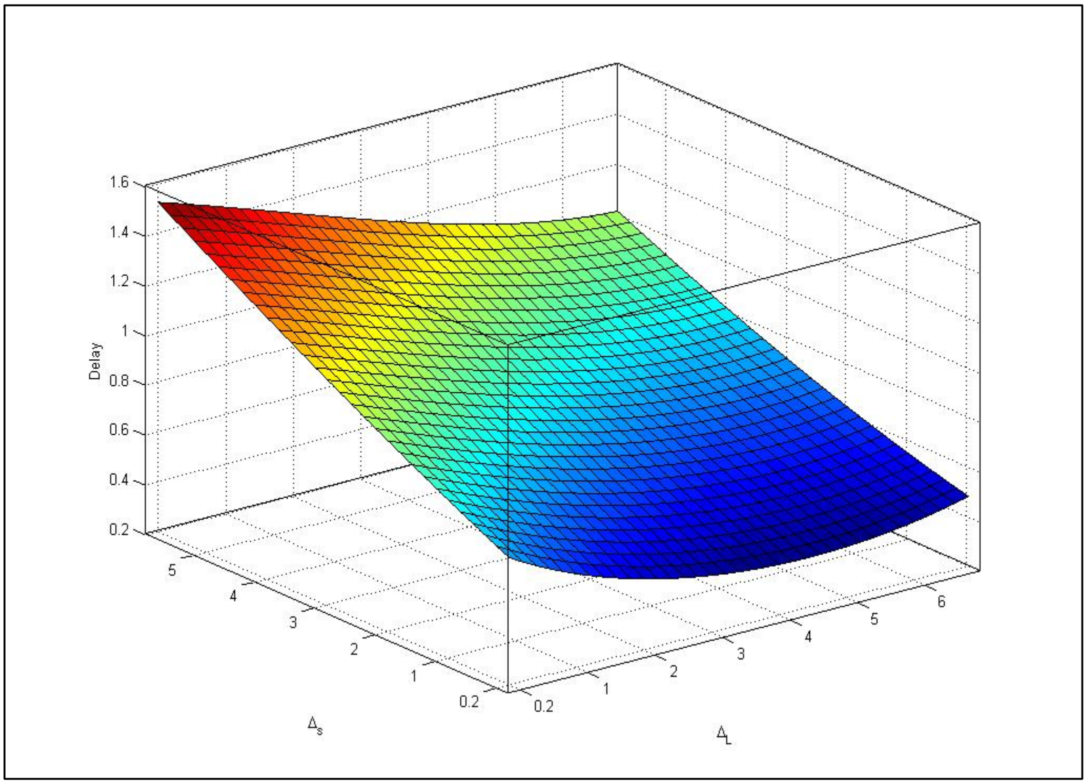

Figure 4: Example average delay per unit time with $\delta = 4.0$ **,** $\lambda_s = 0.02$ **and** $\lambda_L = 0.25$

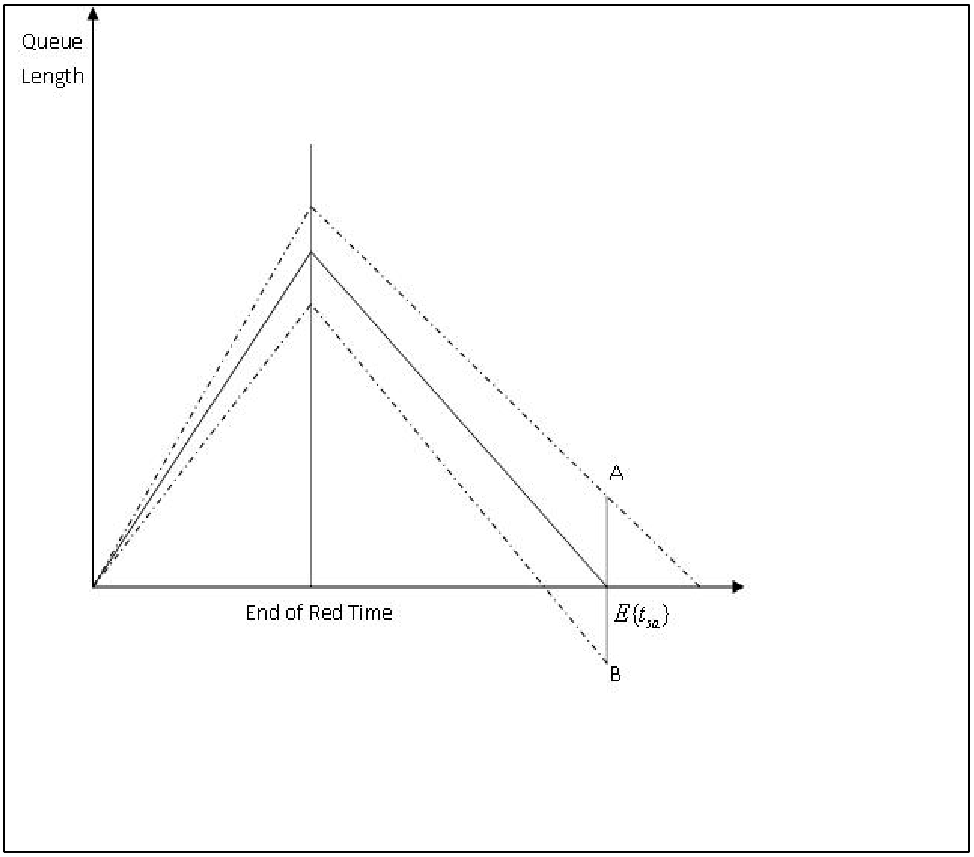

Figure 5: Realizations of cumulative queue