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GAME-THEORETIC ANALYSIS OF DYNAMIC TRAFFIC EQUILIBRIA

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Workability of Asphalt Binders at Mixing Temperatures for Hot and Warm Mix Asphalt

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Abstract

Dynamic traffic assignment has grown steadily in popularity and use since its inception. It has become an important and permanent tool in transportation agencies across the country. However, the exact nature of DTA equilibrium, including existence and uniqueness results, is not fully known in simulation-based models. Specifically, we discuss the possibilities of unrealistic multiple equilibria that arise out of simplified flow models. We demonstrate this with a simple diverge-merge network and discuss piecewise-linear fundamental diagrams as one possible resolution.

Executive Summary

The Dynamic Traffic Assignment (DTA) problem extends the static traffic assignment (STA) problem by considering time-varying travel demand. Given the departure times of travelers in a network, an equilibrium is sought that ensures that travelers between an origin-destination (OD) pair departing at the same time experience equal and minimal travel time. DTA models are advantageous over their STA counterparts as they can capture the effects of dynamic queues and spillback, thus making them more realistic. Dynamic traffic assignment has grown steadily in popularity and use since its inception. It has become an important and permanent tool in transportation agencies across the country.

However, the exact nature of dynamic traffic assignment equilibrium, including existence and uniqueness properties, is still not fully known, especially for simulation-based models. Multiple equilibria are possible, and equilibrium may not exist at all. This presents a problem to practitioners. Traffic flows from the model may be unrealistic and unrepresentative of actual conditions. If planners are using these inaccurate results, their policy decisions can dramatically and permanently worsen the transportation network.

In most dynamic network loading models, oversaturation at a diverge node results in a queue forming on the upstream link, restricting flow to all downstream approaches. When combined with simplified flow models where travel speed is constant for all densities below the critical density, the resulting dynamic equilibrium problem may have infinitely many user equilibrium solutions, despite a unique system optimum solution. We demonstrate this with a simple diverge-merge network, which we also use to show that the price of anarchy in such systems may be unbounded. Our opinion is that this phenomenon is a modeling artifact, rather than something observable in the field. Nevertheless, it poses serious challenges for dynamic traffic assignment algorithms in identifying the “correct” equilibrium solution, and through spillback mechanisms, can propagate unrealistic congestion throughout a network. Simple approaches (such as those based on generating an initial set of paths) may fail to overcome this difficulty, particularly in grid networks and we discuss piecewise-linear fundamental diagrams as one possible resolution.

In light of the computational advantages piecewise linear fundamental diagrams hold, we propose introducing a second piece to the uncongested portion of the fundamental diagram. Adding an additional piece to the fundamental diagram poses no major difficulties to the primary solution methods for the LWR equations, merely adding one additional term to the minimization in the sending flow equation in the cell-transmission or link-transmission

models, or one additional wave speed to check in Newell's method or Daganzo's variational method.

Modifying the fundamental diagram by including a second uncongested linear piece can remedy or limit the effects of this phenomenon without compromising the efficiency of LWR solution methods, even when the resulting fundamental diagrams are only slightly perturbed from a triangular one. Possible directions for future research are also discussed.

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Chapter 1

Introduction

1.1 Dynamic Traffic Assignment Problem

The Dynamic Traffic Assignment (DTA) problem extends the static traffic assignment (STA) problem by considering time-varying travel demand. Given the departure times of travelers in a network, an equilibrium is sought that ensures that travelers between an origin-destination (OD) pair departing at the same time experience equal and minimal travel time. DTA models are advantageous over their STA counterparts as they can capture the effects of dynamic queues and spillback, thus making them more realistic.

DTA models can be classified into two major types: Simulation-based DTA models and analytical DTA models. Simulation-based DTA models use simulation methods to determine the traffic flow propagation on the network; traffic flow dynamics are not solved analytically via a mathematical program. Many DTA models use the LWR theory (Lighthill and Whitham [22], Richards [34]) in the simulation process. However, simulation is only one component of the entire DTA procedure. Modern DTA models involve three linked steps: traffic simulation, path generation, and route flow adjustment as shown in Figure 1.1. The simulation process distributes traffic on the network based on fixed route flows from the previous step. Travel times are calculated, and shortest paths are determined. Path flows are adjusted by shifting some flow on previously used routes to the newly identified shortest paths as to move the solution closer to user equilibrium. Several techniques can be used: Method of Successive Averages (Sheffi and Powell [38]), gradient projection methods (Jayakrishnan et al. [17]), and simplicial decomposition methods (Larsson and Patriksson [20]), among others. Due to the complexity and processes involved in simulation-based DTA models, equilibrium properties have not been proven.

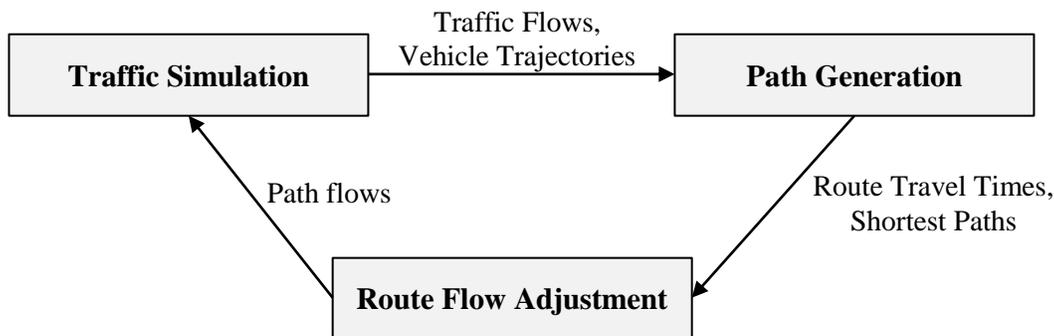


Figure 1.1: The simulation-based dynamic traffic assignment process

Though simulation-based DTA models are the only model type suitable in practice, many analytical methods have also been developed. Analytical models include mathematical programming, optimal control, and variational inequality formulations. Early DTA models, including the first developed by Merchant and Nemhauser [23], were solved using mathematical programs and optimization techniques. Many of these and subsequent models guarantee equilibrium existence, uniqueness, and stability properties (Peeta and Ziliaskopoulos [32]). However, in order to prove these properties, analytical models make restrictive assumptions that hinder the realism of traffic conditions and user behavior. For example, in virtually all mathematical programming methods only one destination can be modeled. If multiple destinations are modeled and the first-in-first-out (FIFO) rule is enforced, the solution constraint space becomes non-convex (Carey [3]). FIFO (i.e., the first vehicle entering a link will be the first vehicle leaving) ensures realistic traffic flows. Other models restrict traffic realism by using static link performance functions (Janson [16]; Boyce et al. [2]) or assuming that route cost functions are decay monotone with respect to route flow (Mounce [25]). Overall, analytical models cannot realistically and efficiently solve large-scale networks and are not used in practice.

1.2 Game Theory and Traffic Assignment

Game theory mathematically models situations where one entities' action interacts or affects another entities' action. It is a broad method that has been applied to a variety of fields: economics, philosophy, resource allocation, biology, political science, military strategy, and network optimization. A game is characterized by three elements: (1) set I consisting of all entities/players, (2) a set of actions/decisions/strategies A_i for every $i \in I$, and (3) the

utility/satisfaction u_i player i will expect from the given set of strategies, $u_i : A \rightarrow \mathbb{R}$.

The most widely used and recognized notion of an equilibrium state or solution of the game theory formulation was developed by John Nash (Nash [26]). It is termed Nash equilibrium and can be categorized into two basic types of equilibrium: pure and mixed strategy. Pure strategy Nash equilibrium can be defined as the stable state where no player can improve his/her utility by changing strategies. It is expressed formally below, where (1) $a_i \in A_i$, (2) a_{-i} indicates the actions of all players except player i , and (3) a'_i indicates all other strategies available to player i besides a_i .

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \forall i \in I$$

A game can have multiple pure strategy Nash equilibria or none at all. However, any game with a finite set of players and a finite set of actions is guaranteed to have a mixed strategy Nash equilibrium as proven by Nash [26]. In a mixed strategy solution, players are allowed to randomize among their various actions – to choose a probability distribution over their strategy set as to maximize their expected utility. Players are assumed to be rational individuals wishing to maximize their payoff and act independently from one another (i.e., players select strategies independently). It is important to note these assumptions are the same assumptions used in the principle of user equilibrium, which all traffic assignment processes are based upon.

The game theory concepts discussed above can be demonstrated using the classic Matching Pennies game. The game consists of two players, Player 1 and Player 2, who each have a penny. Every round, the players choose which side of the penny (heads or tails) to show the other player. If the pennies match sides, Player 1 wins a penny from Player 2. If the pennies do not match, Player 2 wins a penny from Player 1. The Matching Pennies game can be formulated as: (1) $I = \{1, 2\}$, (2) $A_1 = A_2 = \{H, T\}$, and (3) Table 1.1 shows the utilities of each player, where the first element can be interpreted as $(u_1(H, H), u_2(H, H))$.

Table 1.1: Player utilities in the Matching Pennies game

		Player 2	
		Heads	Tails
Player 1	Heads	(+1, -1)	(-1, +1)
	Tails	(-1, +1)	(+1, -1)

As shown in Table 1.1, there is no pure strategy Nash equilibrium. If the pennies match, Player 2 can improve his utility by switching strategies so that the pennies do not match.

However, once this is done, Player 1 can improve his payoff by switching strategies as well, leading to the previous scenario (i.e., Player 2 would switch strategies causing Player 1 to switch strategies, causing Player 2 to switch strategies, causing Player 1 to switch strategies, etc.). There is no stable pure strategy solution; the only equilibrium associated with this problem is a mixed strategy Nash equilibrium. Intuitively, the reader can reason that the equilibrium occurs when both players choose heads/tails 50% of the time. If this weren't the case, e.g., Player 2 shows tails 75% of the time, Player 2 can improve his payoff by lowering his tendency to show tails, eventually reaching the stable point of 50%.

Mixed strategy Nash equilibrium can alternatively be viewed as the solution where players have chosen their probability distributions to make all other players indifferent to their own set of strategies. Applying this methodology to the Matching Pennies game will yield the following set of equations which directly solve for p , the probability of Player 1 choosing heads, and q , the probability of Player 2 choosing heads. Substituting values from Table 1.1, $p = \frac{1}{2}$ and $q = \frac{1}{2}$.

$$p = \frac{u_2(T, T) - u_2(T, H)}{u_2(H, H) + u_2(T, T) - u_2(T, H) - u_2(H, T)}$$

$$q = \frac{u_1(T, T) - u_1(H, T)}{u_1(H, H) + u_1(T, T) - u_1(T, H) - u_1(H, T)}$$

Several research efforts have been conducted regarding static traffic assignment and game theory. These studies will be summarized briefly, since several results are not applicable to DTA or the research in this report. However, the formulation of traffic assignment and the relationship between user equilibrium and Nash equilibrium developed in these papers are crucial.

Charnes and Cooper [4] were the first to discuss the relationship between static traffic assignment equilibrium and Nash equilibrium. They showed that their proposed traffic assignment model satisfied Nash equilibrium conditions. Dafermos [5] proposed a static traffic assignment model for two-way roadways, where link travel time depended on the link volume as well as the volume on the opposite link. Dafermos discussed how a user-optimizing flow pattern in his model satisfied the user equilibrium conditions: all used routes connecting the same origin to the same destination have equal and minimal travel time. She then described how these conditions are equivalent to a Nash equilibrium point.

Devarajan [11] was the first to completely characterize static traffic assignment with continuous flow as an economic game and to mathematically prove that the associated Nash equilibrium corresponds to user equilibrium. Devarajan defined the game as follows: (1) each player $i \in I$ represents a unique origin-destination pair and the set I encompasses all origin-destination pairs in the network, (2) the set of actions for each player includes all routes with feasible network flows connecting the analogous origin-destination pair, and (3) the utility of a player is the resulting total travel time of the sent demand, which the entity tries to minimize. Since the flows are continuous, the players choose from a continuum of pure strategies.

A more intuitive approach was taken by Rosenthal [35]. He formulated traffic assignment as a discrete game, where each player represents an individual driver. The strategies of each player are the set of routes connecting his/her origin to destination. The driver chooses a path to minimize the cost of sending one unit of flow (i.e., to minimize their individual travel time). Rosenthal proves the optimal user equilibrium solution for the discretized version of static traffic assignment corresponds to a pure strategy Nash equilibrium. He also demonstrates that every pure strategy Nash equilibrium does not necessarily satisfy user equilibrium. The following dynamic traffic assignment examples will utilize Rosenthal's game formulation (i.e., drivers are players who choose among their available routes as minimize their travel time) for several reasons: (1) the formulation is simple and intuitive, (2) vehicles are discrete at the fundamental level, and (3) modern dynamic traffic assignment models are discrete in nature, treating vehicles or packets of vehicles as discrete. STA problems also fall under the class of potential games (Monderer and Shapley [24]), which exhibit several interesting mathematical properties. A detailed summary of STA and game theory can be found in Rambha and Boyles [33].

However, modeling the DTA problem as a game is difficult and has not received much attention in literature. The exact nature of dynamic traffic assignment equilibrium, including existence and uniqueness properties, is still not fully known, especially for simulation-based models. Peeta and Ziliaskopoulos [32] warn that the surrounding theory of DTA is underdeveloped, especially regarding the major problem of not providing a universal solution for general networks. Multiple equilibria are possible, and equilibrium may not exist at all. This presents a problem to practitioners. Traffic flows from the model may be unrealistic and unrepresentative of actual conditions. If planners are using these inaccurate results, their policy decisions can dramatically and permanently worsen the transportation network.

In this report, we present several networks that demonstrate the chaotic behavior of DTA equilibrium. One network depicts a scenario with no equilibrium, another with multiple equilibria, and another with infinitely many equilibria. Methods are presented to strengthen the theoretical foundation of DTA and to give planners more confidence in DTA results by interpreting multiple equilibria and guaranteeing that a certain type of equilibrium always exists. Specifically, we discuss the possibilities of unrealistic multiple equilibria that arise out of simplified flow models. Solutions to avoid this are also presented using piecewise linear fundamental diagrams.

Dynamic traffic assignment has grown steadily in popularity and use since its inception. It has become an important and permanent tool in transportation agencies across the country. Improving DTA models and exploring the nature of its equilibrium will give planners more confidence in the accuracy of its results and will further the growth of DTA in previously unexplored sectors.

1.3 Organization of report

The rest of this report is organized as follows: Chapter 2 contains a examples of DTA problems modeled as games. These examples demonstrate the non-existence of equilibria in pure strategies and the possibilities of multiple equilibria. Chapter 3 introduces a modeling artifact that can result in multiple equilibria, most of which are unrealistic. In Chapter 4, we discuss possible ways to resolve this issue using piecewise linear fundamental diagrams.

Chapter 2

Game Theoretic Models for DTA

In this chapter we present two examples, one which demonstrates the non-existence of equilibria in pure strategies and another that has multiple equilibrium solutions. Implications of such phenomena in the process of project selection are also highlighted.

2.1 Non-Existence of Equilibria in Pure Strategies

The game theory formulations presented in this report are meant to display the complicated nature of dynamic traffic assignment (i.e., equilibrium may not exist and multiple equilibria are possible) and propose methods from computational game theory that can address these potential issues. It is a stepping point for future DTA game theory research. Consider the network in Figure 2.1. One vehicle travels from A to B (Vehicle 1) and another from C to D (Vehicle 2). Vertical and horizontal links have a travel time of 1 minute, and diagonal links 1.5 minutes. If two vehicles arrive at a junction simultaneously, priority is determined as indicated in Figure 2.1. The vehicle yielding right-of-way is delayed by an additional minute.

With no interfering traffic, the left path, $A \rightarrow 2 \rightarrow 4 \rightarrow B$, is preferred for traveling from A to B . The bottom path, $C \rightarrow 2 \rightarrow 3 \rightarrow D$, is preferred for traveling from C to D . However, if both travelers choose these paths, Vehicle 1 has priority at node 2 and Vehicle 2 is delayed by an additional minute. Thus, Vehicle 2 opts for the top path, $C \rightarrow 1 \rightarrow 4 \rightarrow 5 \rightarrow D$, with shorter travel time, gaining priority at node 4 and delaying Vehicle 1. Vehicle 1, in turn, opts for the rightmost path, $A \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow B$, which is faster since the second vehicle is on the top path. This frees the way for Vehicle 2 to return to the bottom path, allowing Vehicle 1 to return to the left path regaining priority, and so forth *ad infinitum*. It is clear that there is no deterministic assignment of vehicles to paths that satisfy the equilibrium principle: no matter what combination of routes is chosen one vehicle has a faster alternative.

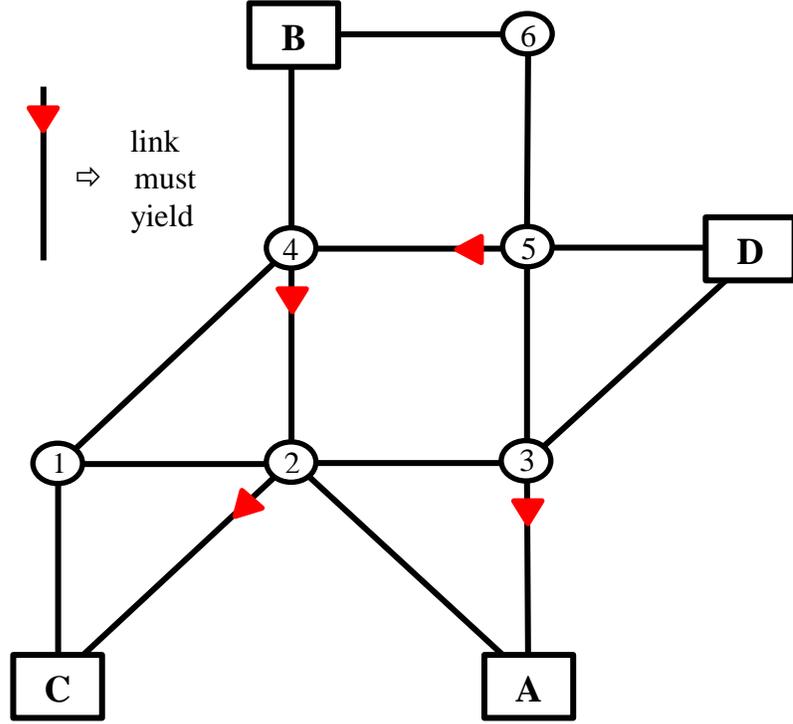


Figure 2.1: Network with no dynamic traffic assignment equilibrium

The situation in Figure 2.1 can be defined as an economic game. $I = \{1, 2\}$. $A_1 = \{\textit{left path (L)}, \textit{right path (R)}\}$, and $A_2 = \{\textit{top path (T)}, \textit{bottom path (U)}\}$. Table 2.1 shows the utilities (travel times) of the travelers.

Table 2.1: Vehicle travel times – no equilibrium example

		Vehicle 2	
		Top	Bottom
Vehicle 1	Left	(4.5, 4.5)	(3.5, 5.0)
	Right	(4.0, 4.5)	(4.0, 4.0)

Analyzing Table 2.1, the reader can confirm no pure strategy Nash equilibrium exists for the game. However, a mixed strategy Nash equilibrium is guaranteed to exist. The probability p of Vehicle 1 choosing the left path and probability q of Vehicle 2 choosing the top path can be determined directly from the equations discussed in the previous chapter.

$$p = \frac{u_2(R, U) - u_2(R, T)}{u_2(L, T) + u_2(R, U) - u_2(R, T) - u_2(L, U)} = \frac{4.0 - 4.5}{4.5 + 4.0 - 4.5 - 5.0} = \frac{1}{2}$$

$$q = \frac{u_1(R, U) - u_1(L, U)}{u_1(L, T) + u_1(R, U) - u_1(R, T) - u_1(L, U)} = \frac{4.0 - 3.5}{4.5 + 4.0 - 4.0 - 3.5} = \frac{1}{2}$$

Therefore, Vehicle 1 will choose the left path 50% of the time, and Vehicle 2 will choose the top path 50% of the time. Since nearly all projects are evaluated at their equilibrium state, an instance that exhibits no equilibrium is an issue for transportation planners. There would be no equilibrium state to analyze, no point of comparison among other project alternatives. Also, if the stopping criterion of the DTA model is lax, project analysis will occur at a state of transition – which may be an inaccurate representation of the true traffic condition. However, by formulating the no equilibrium example as a game more information is provided to the practitioner. The p and q values can be interpreted as the fixed ratio of travelers choosing a certain path (among a set of paths) from their respective origin to destination. For example, 50% of users departing from A to B will use the left path, while 50% of users traveling from C to D will use the top path. With this information planners can approximate traffic flows on the network within a given time period, which can be used to estimate other indices of interest (e.g., crash rates, pavement deterioration, required maintenance, etc.).

2.2 An example with multiple equilibria

Figure 2.2 depicts a network with multiple dynamic traffic assignment equilibrium. Vehicle 1 travels from A to B , and Vehicle 2 travels from C to D . The travel time on all links is one minute. As indicated in Figure 2.2, if Vehicle 1 and Vehicle 2 arrive at node 3 simultaneously, Vehicle 1 must yield and will incur one minute of extra delay. Links $2 \rightarrow 1$ and $1 \rightarrow 4$ are directed arcs (i.e., one-way streets).

Vehicle 2 has two shortest paths: (1) the top path, $C \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow D$ and (2) the bottom path, $C \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow D$. Both paths have 4 minute travel times and are independent of Vehicle 1's actions. Vehicle 1 has two plausible paths: (1) the left path, $A \rightarrow 1 \rightarrow 3 \rightarrow B$ and (2) the rightmost path, $A \rightarrow 1 \rightarrow 4 \rightarrow 3 \rightarrow B$. The path travel times are influenced by Vehicle 2. The travel time of the left path will be 3 minutes if Vehicle 2 chooses the bottom path and 4 minutes if the top path. There are three user equilibrium solutions associated with this example: (1) Vehicle 1 chooses the left path and Vehicle 2 chooses the top path, (2) Vehicle 1 chooses the left path and Vehicle 2 chooses the bottom path, and (3) Vehicle 1 chooses the rightmost path and Vehicle 2 chooses the bottom path. They are bolded in Table 2.2.

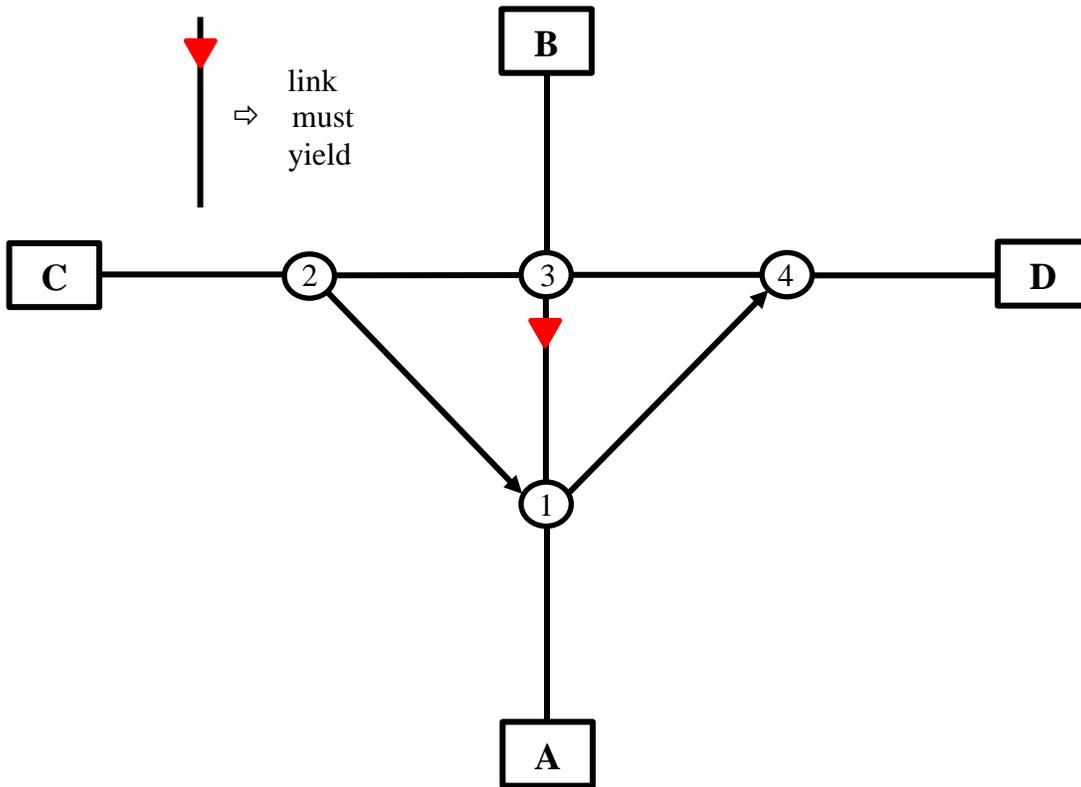


Figure 2.2: Network with multiple dynamic traffic assignment equilibria

Table 2.2: Vehicle travel times – multiple equilibria example

		Vehicle 2	
		Top	Bottom
Vehicle 1	Left	(4, 4)	(3, 4)
	Right	(4, 4)	(4, 4)

As shown in Table 2.2, the bolded elements also correspond to pure strategy Nash equilibrium points. However, applying equilibrium refinements from the game theory literature can reduce the number of unrealistic equilibrium solutions. One such refinement is the concept of trembling hand perfect equilibrium introduced by Selten [37]. The trembling hand property is based on the notion that players may, through “a slip of the hand”, choose an incorrect strategy (i.e., players may not choose the strategy predicted by Nash equilibrium 100% of the time). Loosely, the definition of trembling hand perfect equilibrium is: a pair of pure strategies (i, j) is a trembling hand perfect equilibrium if and only if i is a best reaction for Player 1 not only to the pure strategy j of Player 2 (which it is in Nash equilibrium), but

to mixed strategies in which Player 2 plays each of his pure strategies with some positive probability, and vice versa for Player 2.

Employing the trembling hand concept, it only makes sense for Vehicle 1 to choose the rightmost path if Vehicle 2 chooses the top path 100% of the time; the travel time on the left path will always be less than or equal to the travel time on the rightmost path. The pure strategy Nash equilibrium where Vehicle 1 chooses the rightmost path and Vehicle 2 chooses the top path is not a trembling hand perfect equilibrium. Therefore, the associated user equilibrium can be eliminated, and the amount of multiple DTA equilibria is reduced. This reduction is vitally beneficial to transportation engineers since any collected data, any comparison, any measured index must be made from a network model that represents realistic traffic conditions.

2.3 Project selection in the presence of multiple equilibria

Assume a transportation planning agency is evaluating a set of project alternatives. Their goal is to minimize the total system travel time (TSTT) given a budget M . The base scenario is similar to the multiple equilibria network in Figure 2.2, except the travel times of the directed links are 1.5 minutes. These conditions yield a single user equilibrium: Vehicle 1 will use the left path, $A \rightarrow 1 \rightarrow 3 \rightarrow B$, and Vehicle 2 will use the top path, $C \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow D$. The total travel time of the system is 8 vehicle-minutes. Project 1 is the same as the base case except the travel time on link $4 \rightarrow D$ has been reduced to 0.5 minutes. The equilibrium path set of Project 1 will be the same as the base case (i.e., Vehicle 1 will use the left path, Vehicle 2 will use the top path); however, TSTT is 7.5 vehicle-minutes. The cost of implementing Project 1 is M . Project 2 is the multiple user equilibria network discussed earlier, the improvements being the directed arcs have a reduced travel time of 1.0 minutes. The cost of Project 2 is also M .

Using the refined pure strategy Nash equilibrium solutions via the trembling hand theory, there are two distinct user equilibria associated with Project 2: (1) Equilibrium I – Vehicle 1 will choose the left path, and Vehicle 2 will choose the top path, and (2) Equilibrium II – Vehicle 1 will choose the left path, and Vehicle 2 will choose the bottom path. Equilibrium I will have a TSTT of 8 vehicle-minutes, while Equilibrium II will have a TSTT of 7 vehicle-minutes. The project alternatives are summarized in Table 2.3.

Table 2.3: Project alternatives

Network	Link travel time modifications	Cost	Equilibrium paths	TSTT (veh-min)
Base Case	-	-	Vehicle 1: Left Vehicle 2: Top	8
Project 1	Link 4 \rightarrow D: 0.5 minutes	M	Vehicle 1: Left Vehicle 2: Top	7.5
Project 2	Links 2 \rightarrow 1, 1 \rightarrow 4: 1.0 minutes	M	Vehicle 1: Left Vehicle 2: Top Vehicle 1: Left Vehicle 2: Bottom	8 7

It is nearly impossible to detect multiple dynamic traffic assignment equilibria in current software programs. Practitioners are simply given an approximate solution. Therefore, depending on the user equilibrium outputted from the DTA model, different projects will be selected. If the model converges to Equilibrium I, Project 1 will be selected. If Equilibrium II is generated by the model, Project 2 will be selected. As the example shows, multiple DTA equilibria can have a dramatic impact on planning decisions and thusly, major (and potentially permanently negative) impacts on the transportation system. For example, suppose Equilibrium II represents the actual traffic condition and Equilibrium I was outputted from the model. The budget M would be spent on implementing Project 1. Funds would be spent half as efficiently, and the travel time improvements would only be felt by Vehicle 2.

Chapter 3

Modeling Issues

In most dynamic network loading models, oversaturation at a diverge node results in a queue forming on the upstream link, restricting flow to all downstream approaches. When combined with simplified flow models where travel speed is constant for all densities below the critical density, the resulting dynamic equilibrium problem may have infinitely many user equilibrium solutions, despite a unique system optimum solution. We demonstrate this with a simple diverge-merge network, which we also use to show that the price of anarchy in such systems may be unbounded. We feel that this issue is an artifact of modeling assumptions, rather than a description of a phenomenon in the field, and discuss piecewise-linear fundamental diagrams as one possible resolution.

3.1 Introduction

This chapter discusses the implications of two common assumptions in dynamic network loading models: (1) queues at diverge nodes obstruct traffic to all downstream links; and (2) vehicles travel at free-flow speed whenever vehicle density is subcritical. These assumptions underlie many of the large-scale DTA models used in practice today, greatly simplifying the calculations involved. However, we present a small example in which, under these two assumptions, literally every feasible route assignment is a user equilibrium, although these solutions correspond to vastly different flow conditions and the system optimum solution is unique. In passing, we show that the price of anarchy in such systems can be arbitrarily high.

The setting for our examples is a basic diverge-merge network (Figure 3.1), similar to the ones used by Daganzo [8] and Nie [30] to discuss other complications in dynamic network loading. Daganzo uses this network to show that these models can exhibit chaotic behavior due to

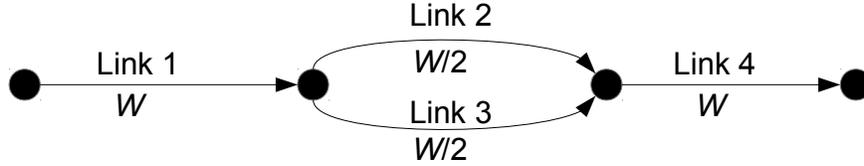


Figure 3.1: Network with link capacities shown.

queue spillback, and in particular that increasing the capacity on a “bottleneck” link may worsen conditions due to self-optimizing drivers choosing shortest routes. Nie demonstrates that the user equilibrium solutions in such a network are not unique, and proposes stability and efficiency criteria for distinguishing among these equilibria. In fact, the equilibria we study in this chapter were briefly noted by Nie [30, Equilibria IV in Section 3], but the focus of his paper was on the other, “efficient” equilibria. In this report, we perform a deeper analysis of the “inefficient” equilibria. In contrast to both Daganzo [8] and Nie [30], which focused on the effects of congestion arising at a merge node, we consider congestion arising at the *diverge* node: in our examples, the only network congestion will occur upstream of the diverge node.

As discussed in more detail below, our concern is not that this phenomenon occurs in practice, but that it may occur during solution of DTA models, giving the illusion of much greater congestion than would exist in reality. Furthermore, this phenomenon may be difficult to detect in large networks. Modifying the fundamental diagram to include more than one linear piece for subcritical densities can reduce the impact of this phenomenon without significantly complicating the network loading. Our intent in this report is primarily to call attention to this possibility, and to spur further research into its prevalence and potential resolutions.

We first describe the network loading model and assignment logic, presents the basic example, and remarks on the price of anarchy and then discuss the practical implications of our example, and how it may manifest in larger networks.

3.2 Primary Results

3.2.1 Network model

The phenomenon raised in this chapter arises under rather general assumptions, to be discussed in more detail in Chapter 4. However, for concreteness, we describe our example based on the LWR hydrodynamic theory Lighthill and Whitham [22], Richards [34], which

postulates (1) that the rate of flow q at any point and time is a function of density k alone — that is, $q(x, t) = Q_{x,t}(k(x, t))$ at any location x and time t ; (2) that the conservation law $\partial q/\partial x + \partial k/\partial t = 0$ holds everywhere these derivatives exist; and (3) that the space-mean speed u is related to q and k by the *fundamental equation* $q = uk$. The function $Q_{x,t}$ is often called the *fundamental diagram*. Given appropriate boundary conditions, this network loading model can be formulated as the solution to a hyperbolic partial differential equation. The solution of this model is greatly simplified by assuming a piecewise-linear fundamental diagram; under this assumption, the solution is readily obtained by the method of characteristics Newell [27, 28, 29], a finite difference scheme Daganzo [6, 7], or through the solution of a shortest-path problem Daganzo [9, 10].

Omitting the location and time indices for brevity, the fundamental diagram $Q(k)$ is continuous, concave, and has two zeros: one at $k = 0$ and the other at the *jam density* $k = k_j$. We define the *capacity* $\bar{q} = \max_k Q(k)$, and the *critical density* $k_c = \min\{k : Q(k) = \bar{q}\}$. A density value k is *subcritical* if $k < k_c$. In the case of a triangular fundamental diagram (Figure 3.2), $Q(k)$ can be completely specified by any three of the following four parameters: the free-flow speed v , the backward wave speed $-w$, the capacity \bar{q} , and the jam density k_j . It is convenient to write $Q(k) = \min\{S(k), R(k)\}$ in terms of the *sending flow* $S(k) = \min\{kv, \bar{q}\}$ and *receiving flow* $R(k) = \min\{w(k_j - k), \bar{q}\}$ in this case.¹ Other piecewise linear fundamental diagrams have been proposed in the literature; for instance, the trapezoidal diagram in Daganzo [6] is commonly used, with a horizontal piece corresponding to the capacity \bar{q} . For brevity we initially assume a triangular fundamental diagram of two pieces; however, in all examples in this report density remains subcritical, so need only concern ourselves with the increasing portion of the fundamental diagram, and the remarks in this report would apply equally to trapezoidal fundamental diagrams or any other with a single increasing linear piece.

In the network in Figure 3.1 there are four links with unit length; relative to each link, $x = 0$ denotes the upstream end and $x = 1$ the downstream end. We assume that the fundamental diagram is homogeneous on each link and constant with time, using \bar{q}_i to refer to the capacity at each point on link i , k_j^i the jam density on link i , and so forth. At the diverge, we are interested in the inflow rates r_2 and r_3 from the downstream end of link 1 to the upstream ends of links 2 and 3, respectively. Defining the sending and receiving flows at location x on link i at time t as $S_i(x, t)$ and $R_i(x, t)$, r_2 and r_3 depend on $S_1(1, t)$, $R_2(0, t)$, and $R_3(0, t)$.

¹We describe the problem in continuous time; discrete equivalents of these formulas can be found in Daganzo [7], Yperman [42], and Nie et al. [31].

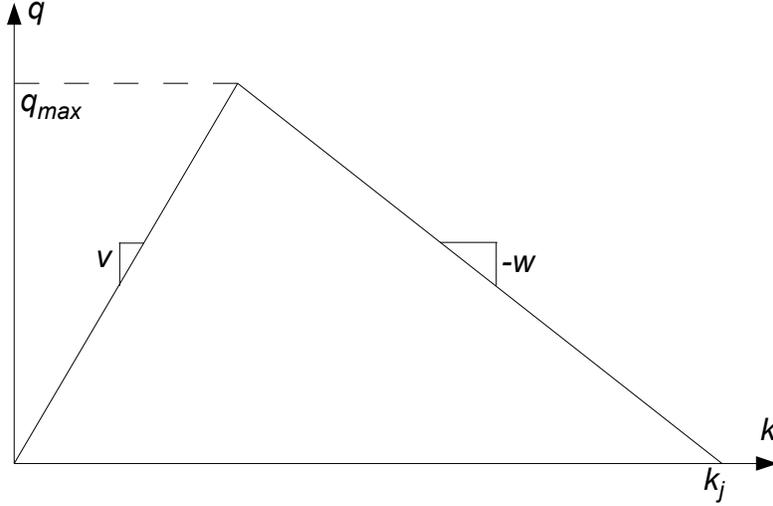


Figure 3.2: Triangular fundamental diagram specified by any three of the four parameters shown.

Using $p_{1i}(t)$ to denote the fraction of the sending flow wishing to travel to link i ($i \in \{2, 3\}$), we have $r_i(t) = \phi(t)p_{1i}(t)S_1(1, t)$ where $\phi(t) = \min_{i \in \{2, 3\}} \{1, R_i(0, t)/[p_{1i}(t)S_1(1, t)]\}$. Notice that if one of the downstream links i' is oversaturated — that is, if $R_{i'}(0, t) < p_{1i'}(t)S_1(1, t)$ — the flow to *both* links 2 and 3 is restricted.

A variety of merge models exist (Daganzo [7], Lebacque [21], Jin and Zhang [18], Ukkusuri et al. [40]). However, in all of the examples in this report, we assume $\bar{q}_4 \geq \bar{q}_2 + \bar{q}_3$, so there is no congestion at the merge node and we only need to specify the trivial case of no merge congestion: $q_{i4}(t) = S_i(1, t)$ for all t and $i \in \{2, 3\}$.

We assume a total of D vehicles enter the upstream end of link 1 at a uniform rate W . (That is, the inflow rate is W for $t \in [0, D/W]$ and zero afterward.) Route choice is often expressed in terms of the rate at which vehicles depart on each path in the network. However, in our network, it is simpler to express route choice solely through the splitting proportions at the diverge. Since $p_{12}(t)$ and $p_{13}(t)$ are nonnegative and sum to one, we can use the single parameter $p(t) \in [0, 1]$ to describe route choice at each point in time, with $p_{12}(t) = p(t)$ and $p_{13}(t) = 1 - p(t)$. Expressing route choice in this way simplifies the calculations and is equivalent to specifying path departure rates under mild regularity assumptions; for instance, it is sufficient to assume path departure rates and $p(t)$ are continuous almost everywhere, an assumption we adopt. We are interested in the functions $p(t)$ which create dynamic user equilibria according to Wardrop's principle (1952): that no traveler may reduce his or her

travel time by unilaterally switching routes. In a continuous-flow setting, this implies that for any departure time, the travel times on all used paths are equal and minimal.

In our example, we let $\bar{q}_2 = \bar{q}_3 = W/2$, $\bar{q}_1 = \bar{q}_4 = W$, $v^i = 3$ on all links (so the free-flow time through the network is 1), and $k_j^i = D$ on all links (so there will be no queue spillback). With this selection of parameters, the only place congestion could possibly occur is upstream of the diverge node. Since the route choice $p_{12}(t)$ is only made at the diverge node itself (the downstream end of any queue on link 1), it should be clear that any possible function $p(t)$ results in a user equilibrium, because all links downstream of the diverge node will be at free-flow. Different $p(t)$ functions will create different amounts of delay upstream of the queue, but this delay never affects the vehicle at the front of the queue, which is the only place where route choice can be exercised. Since the user equilibrium assumption is only concerned with a driver's own delay, and never that of other drivers in the system, it follows that any choice of $p(t)$ satisfies the user equilibrium condition. The following subsections explore this issue quantitatively, respectively deriving the same result by direct calculation of delays, modifying the network to illustrate an unbounded price of anarchy, and conducting a broader sensitivity analysis with less contrived parameters.

3.2.2 Delay calculations

The previous subsection argued that any choice of $p(t)$ produces a user equilibrium solution, due to the selfish nature of equilibrium routing and the network structure. This subsection verifies this logic by writing expressions for the travel time on each path given $p(t)$, which will also serve as the basis for additional analysis.

As stated above, the only possible congestion in the network occurs at the downstream end of link 1. The diverge model ensures that the inflow rates to links 2 and 3 never exceed their capacity, and since the capacity of link 4 is the sum of the capacities of links 2 and 3, no congestion will occur at the merge. Until all D vehicles exit link 1, its outflow rate will be $W\phi(t)$. Therefore, the cumulative number of vehicles which have exited link 1 by time t is given by

$$N_1^\downarrow(t) = \int_0^t S_1(t')\phi(t') dt' \quad (3.1)$$

while the cumulative number of vehicles that have entered link 1 by time t is $N_1^\uparrow(t) = Wt$. Let T be the time at which the last vehicle exits link 1, that is, $N_1^\downarrow(T) = D$. Substituting

the parameters of the network model into the diverge equation, we have

$$\phi(t) = \min\{1/[2p(t)], 1/[2(1-p(t))]\} \quad (3.2)$$

for $t \in (1/3, T)$ since there is no spillback from links 2 and 3, implying $R_2(0, t) = R_3(0, t) = W/2$.

Furthermore, $S_1(t) > 0$ and $\phi(t) > 0$ for $t \in (1/3, T)$ imply that $N_1^\uparrow(t)$ and $N_1^\downarrow(t)$ are strictly increasing and thus have well-defined inverse functions $t_1^\uparrow(n)$ and $t_1^\downarrow(n)$, respectively denoting the time at which the n -th vehicle enters and exits link 1. Thus, for $n \in (0, D)$, the travel time of the n -th vehicle on link 1 is given by

$$\tau_1(n) = t_1^\downarrow(n) - t_1^\uparrow(n) \quad (3.3)$$

Since both paths in the network consist of three links, two of which are never congested and have free-flow time $1/3$, the total travel time of the n -th vehicle is

$$\tau(n) = t_1^\downarrow(n) - t_1^\uparrow(n) + 2/3 \quad (3.4)$$

In particular, $\tau(n)$ *does not depend on the path chosen* — for all vehicles, the travel time is the same on either path available to them, so *all* functions $p(t)$ produce solutions satisfying the Wardrop user equilibrium conditions. However, as seen in the next section, these equilibria are *not* equivalent in terms of experienced travel time.

3.2.3 Price of anarchy

It is common to distinguish among multiple equilibria in terms of the total travel time of all vehicles in the system; in particular, following Koutsoupias and Papadimitriou [19], one can define the “price of anarchy” ρ of a system to be the ratio between the *worst* user equilibrium (in terms of total travel time) and the system-optimal solution which minimizes total travel time (which need not be a user equilibrium). In this subsection, we show that by varying the parameters in our example, the price of anarchy can be arbitrarily high. This is in contrast to many other network equilibrium problems, in which the price of anarchy can be bounded: Roughgarden [36] show that $\rho \leq 4/3$ for a static equilibrium problem with affine cost functions, and that $\rho = \Theta(p/\log p)$ for polynomial cost functions of degree p . Explicit bounds have been found for other equilibrium variants, including static equilibrium with congestion pricing Han and Yang [14], static stochastic user equilibrium with logit disturbance terms Guo et al. [13], Huang et al. [15], and dynamic single-bottleneck models Doan

and Ukkusuri [12]. Anshelevich and Ukkusuri [1] showed that the price of anarchy can be arbitrarily large in dynamic networks; however, in their example, the price of anarchy only grew with the network size. Below, we show that the price of anarchy can be made large even with a four-link network.

Define the total cost of travel to be

$$\Sigma = \int_0^D \tau(n) dn \quad (3.5)$$

Clearly $\tau(n) \geq 1$ for all n (since that is the free-flow travel time). This lower bound can in fact be attained for all vehicles: $t_1^\uparrow(n)$ is simply n/W , and if $p(t) = 1/2$ for all t , $\phi(t) = 1$, $S_1(t) = W$ for $t \in [1/3, T]$, so

$$N_1^\downarrow(t) = \int_0^t S_1(t') \phi(t') dt' = \begin{cases} 0 & \text{if } t < 1/3 \\ W(t - 1/3) & \text{if } t \in [1/3, T] \\ D & \text{if } t > T \end{cases} \quad (3.6)$$

$N_1^\downarrow(t)$ must be continuous, so we have $T = 1/3 + D/W$, and can calculate the inverse function $t_1^\downarrow(n) = 1/3 + n/W$ for $n \in (0, D)$; hence $\tau_1(n) = (1/3 + n/W) - n/W = 1/3$ and $\tau(n) = 1$, so $\Sigma = D$ in this system-optimal solution.

To find an upper bound on Σ , we need to choose $p(t)$ to maximize $\tau_1^\downarrow(n)$ for all n . While in general such a maximization problem would require techniques from the calculus of variations, this example can be solved through a simpler argument: maximizing $\tau_1^\downarrow(n)$ is equivalent to maximizing $t_1^\downarrow(n)$ for all n (since $t_1^\uparrow(n)$ is given), which is equivalent to minimizing $N_1^\downarrow(t)$ for all t , which is equivalent to minimizing $\phi(t)$ for all t , which occurs if $p(t) = 1$ or 0 for all $t \in (0, T)$. Performing the calculations in this case, we have

$$T = 1/3 + 2D/W \quad (3.7)$$

$$N_1^\downarrow(t) = W(t - 1/3)/2 \text{ for } t \in [1/3, T] \quad (3.8)$$

$$t_1^\downarrow(n) = 1/3 + 2n/W \quad (3.9)$$

so finally

$$\tau_1(n) = 1/3 + n/W \quad (3.10)$$

$$\tau(n) = 1 + n/W \quad (3.11)$$

and

$$\Sigma = D + D^2/2W \quad (3.12)$$

Since all solutions are user equilibria, we may define the price of anarchy ρ to be the ratio between these upper and lower bounds on Σ found above:

$$\rho = \frac{D + D^2/2W}{D} = 1 + \frac{D}{2W} \quad (3.13)$$

which can clearly be made arbitrary large by increasing the ratio D/W .

3.2.4 Sensitivity analysis

This section discusses how the above analysis would change under different choices of parameters. The most important parameters in our example are \bar{q}_2 , \bar{q}_3 , and the inflow rate W to the network. The choice of \bar{q}_4 and \bar{q}_1 are less significant as long as $\bar{q}_4 \geq \bar{q}_2 + \bar{q}_3$ and $\bar{q}_1 \geq W$, assumptions we retain as essential features of our demonstration. In particular, we allow $\bar{q}_2 \neq \bar{q}_3$, but enforce $\bar{q}_2 \leq \bar{q}_3$ by symmetry. Allowing W , \bar{q}_2 , and \bar{q}_3 to take generic positive values within these assumptions, we divide the analysis into three major cases:

Case I: $W < \bar{q}_2$. In this case, no congestion can arise at the diverge node, since $\phi(t) = 1$ regardless of $p(t)$, and all solutions have $\Sigma = D$. Therefore all solutions are both user equilibria and system optima, and $\rho = 1$.

Case II: $W > \bar{q}_2$ but $W \leq \bar{q}_2 + \bar{q}_3$. In this case, it is possible to avoid congestion by suitably dividing the outflow from link 1 between links 2 and 3, so in the system-optimum solution all vehicles experience unit delay and $\Sigma = D$. However, congestion can be caused if p is too large: if $p(t) > \bar{q}_2/W$, $\phi(t) < 1$, but such solutions remain user equilibria because the only source of delay is upstream of the diverge node. Repeating the analysis from the previous subsection, we find that the worst-case delay is obtained with $p(t) = 1$ for all t , for which

$$\Sigma = D + \frac{D^2}{2} \left(\frac{1}{\bar{q}_2} - \frac{1}{W} \right) \quad (3.14)$$

and

$$\rho = 1 + \frac{D}{2} \left(\frac{1}{\bar{q}_2} - \frac{1}{W} \right) \quad (3.15)$$

and again the price of anarchy can be made arbitrarily large by increasing D . (The previous subsection's analysis is a special case.)

Case III: $W > \bar{q}_2 + \bar{q}_3$. In this case, some congestion is inevitable, since links 2 and 3 do not have enough capacity to handle the outflow from link 1. Delay is minimized by maximizing $\phi(t)$, which occurs if $\bar{q}_2/p(t) = \bar{q}_3/(1-p(t))$ or $p(t) = \bar{q}_2/(\bar{q}_2 + \bar{q}_3)$. In this case, the resulting total delay is

$$\Sigma = D + \frac{D^2}{2} \left(\frac{1}{\bar{q}_2 + \bar{q}_3} - \frac{1}{W} \right) \quad (3.16)$$

as the reader may verify. As with Case II, delay is maximized with $p(t) = 1$ for all t , resulting in

$$\Sigma = D + \frac{D^2}{2} \left(\frac{1}{\bar{q}_2} - \frac{1}{W} \right) \quad (3.17)$$

As before, all solutions are user equilibria, but the system optimum is unique ($p(t) = \bar{q}_2/(\bar{q}_2 + \bar{q}_3)$). Therefore, the price of anarchy is

$$\rho = \frac{1 + D(1/\bar{q}_2 - 1/W)/2}{1 + D(1/(\bar{q}_2 + \bar{q}_3) - 1/W)/2} \quad (3.18)$$

which is bounded for any fixed values of \bar{q}_2 and \bar{q}_3 even as $D \rightarrow \infty$, but which increases without bound if $\bar{q}_2 + \bar{q}_3 \rightarrow W$ at the same time as $D \rightarrow \infty$, again yielding the results of the previous subsection in the limit.

To summarize, regardless of the capacity values chosen, all feasible solutions are user equilibria if there is no downstream congestion at the merge; the implications in terms of the delay depend on the relative values of the upstream link outflow rate and the downstream links' capacities. If the inflow rate is sufficiently small, no congestion will occur and all feasible solutions are system optima; if the inflow rate is sufficiently large, a certain level of congestion is inevitable, but the system optimal solution is unique and the price of anarchy is strictly greater than one. For inflow rates between these extremes, some user equilibria produce congestion while others do not, and the price of anarchy can thus be made arbitrarily high by manipulating the total demand.

Chapter 4

Discussion

The previous chapter demonstrated that simple dynamic diverge-merge networks exist where every feasible path assignment is a user equilibrium, despite major differences in total travel time as path assignments vary. In this Chapter, we discuss the implications of these findings, addressing three questions in turn: whether this phenomenon is likely to exist in reality, or simply a modeling artifact; the implications on DTA modeling and algorithm termination; and whether this phenomenon would be localized, or if impacts could be felt throughout a larger network.

Regarding the first question, we believe this phenomenon is unlikely to occur in the field for several reasons. First, from entropy considerations, if both paths have equal travel times, it is improbable that all travelers would only select one path. Further, the previous results rely heavily on the modeling assumption that travel times will continue to be equal on both paths despite unequal usage, which follows from the assumption of the triangular fundamental diagram. In practice, due to heterogeneity in driver behavior, average speed will drop even at lower densities if overtaking is prohibited, and a travel time-minimizing driver would probably choose the link with lesser density to maximize his or her probability of driving at the preferred speed. If this is the case, the unique equilibrium occurs with $p(t) = 1/2$ for all t . In Section 4.1, we return to the fundamental diagram as one possible resolution of the issue.

However, even if this phenomenon never occurs in the field, it poses great challenges from a modeling or algorithmic perspective. Given the example in the previous chapter, any DTA software using a gap criterion to terminate will stop after the first iteration, since the user equilibrium principle is satisfied exactly. If this first iteration is an all-or-nothing assignment, the reported level of congestion will be the highest possible value, even though the more likely

field conditions would be the lowest possible value. Even continuing a DTA algorithm for another iteration after such “convergence” is obtained may not detect the second, unused, path: if ties are broken deterministically in the time-dependent shortest path computation, many simple path generation routines will return the first path again since the travel times remain at free-flow on both downstream segments. While it may be possible to check for the presence of this phenomenon in small networks, detecting this phenomenon in larger networks can be more problematic. Furthermore, this issue is also not confined to the LWR model and its variations; any flow model in which free-flow speeds are maintained even as density rises is subject to the same analysis, such as point queue models.

Still worse, this phenomenon can propagate through larger networks. In the example above, link jam densities were chosen to avoid any spillback; however, if this phenomenon were to persist over time, spillback on the upstream link would likely occur, reducing capacity on additional upstream links. In a grid network (Figure 4.1), each block presents an alternative set of choices roughly similar to a diverge and merge. While not directly identical, due to the presence of intermediate nodes and the differences between diverge models and general intersections, congestion upstream of intersections combined with free-flow conditions downstream will still produce the same result. When modeling networks with high levels of congestion, even if this phenomenon occurs only at a small handful of blocks in a central business district, the resulting spillback effects could result in a huge increase in congestion.

One potential way to mitigate this modeling effect is to initially generate a working set of multiple paths per origin-destination pair, perhaps using a K-shortest path algorithm. This approach is often employed to avoid wasting early DTA iterations on highly congested solutions based on all-or-nothing assignments, and one may hope that this strategy may also avoid the worst of the equilibria discussed here. While studying this strategy is beyond the scope of this report, we speculate that its impact may be more limited than one would hope. In grid networks, the number of equal-length paths between two points is quite large, and missing even one such path may trigger spillback and obstruction of other links and origin-destination pairs.

The root causes of the phenomenon described in this report are queuing located upstream of diverges, and free-flow conditions downstream (even with unequal link utilizations). This suggests two approaches to resolving the issue and the difficulties it imposes on DTA models in determining convergence. One alternative is to modify the diverge model, perhaps providing separate queues for travelers heading for different links, as with independent turn

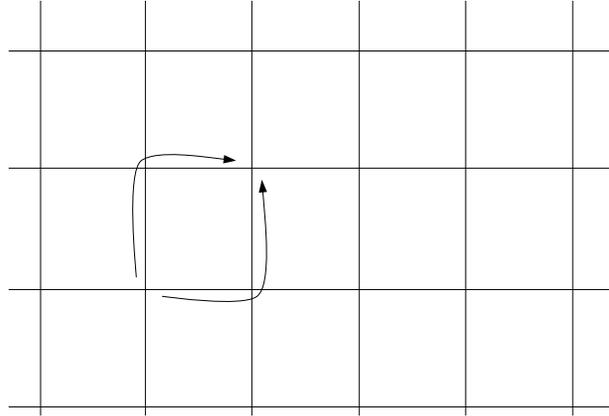


Figure 4.1: Grid networks have roughly similar topology to the diverge-merge network.

lanes. However, this only pushes the problem further upstream: it does not make sense for the capacity for each of these separate queues to exceed the capacity for the downstream links, and effectively the diverge point has been moved to the point at which the turn lanes split, rather than the point at which the links themselves split.

However, the second cause admits a more promising solution: adjusting the fundamental diagram so that speeds drop below free-flow even at subcritical densities. This technique is discussed in detail in the next section, where we show that equilibria can be made unique (except for an arbitrarily small interval of low demand) with an arbitrarily small perturbation of travel times. While the Highway Capacity Manual (HCM) Transportation Research Board [39, Exhibit 11-6] suggests travel times begin to decrease when density exceeds 1000-2000 passenger cars per lane, we actually propose introducing a perturbation at a much smaller density value. We believe this distinction is acceptable for the following reason. The HCM and DTA serve different purposes — if average speed is nearly constant with density over a particular range, for the purposes of operational analysis there can be considerable advantage in approximating the speed as constant, simplifying the model and avoiding the impression of false precision. However, as we have shown in this report, in DTA this assumption can produce an infinite number of equilibria and an unbounded price of anarchy. Perturbing the fundamental diagram can address this issue while introducing a change in travel times which can be made as small as desired, as we show in the next section. We feel the benefits of reducing the size of the equilibrium set outweigh the minor loss in accuracy in travel time calculation.

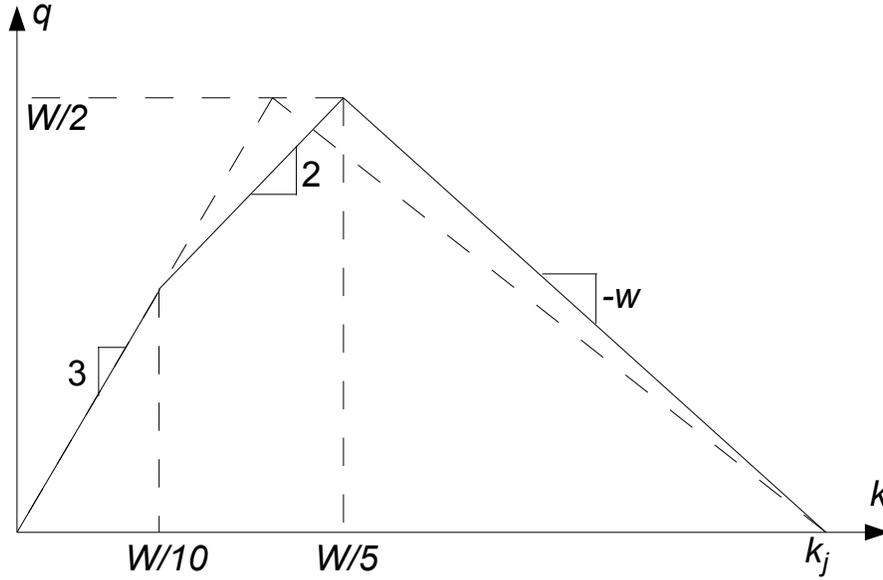


Figure 4.2: Piecewise linear fundamental diagram for links 2 and 3; capacity and jam density unchanged.

4.1 Piecewise-linear fundamental diagrams

In light of the computational advantages piecewise linear fundamental diagrams hold, we propose introducing a second piece to the uncongested portion of the fundamental diagram, as in Figure 4.2. Adding an additional piece to the fundamental diagram poses no major difficulties to the primary solution methods for the LWR equations, merely adding one additional term to the minimization in the sending flow equation in the cell-transmission or link-transmission models, or one additional wave speed to check in Newell’s method or Daganzo’s variational method.

We demonstrate this by modifying the previous example in this way, beginning with an illustrative example before addressing more fully the question of how the additional piece should be added. Figure 4.2 shows a piecewise linear fundamental diagram for links 2 and 3, with an added line segment in the uncongested region of slope 2, with the original triangular fundamental diagram superimposed. Note that the critical density and backward wave speed have changed in order to keep the capacity and jam density unchanged; specifically, the critical density increased from $W/6$ to $W/5$. For this diagram, link speeds are decreasing in k whenever $k > W/10$, with a unique speed corresponding to each such density value. When the steady inflow rate to link i is r_i with this fundamental diagram, a routine application of

Newell's method yields the relation

$$N_i^\downarrow(t) = \min\{r_i t - r_i/2 + W/20, r_i t - r_i/3\} \quad (4.1)$$

where $i \in \{2, 3\}$ or

$$N_i^\downarrow(t) = \begin{cases} r_i t - r_i/2 + W/20 & \text{if } r_i \geq 3W/10 \\ r_i t - r_i/3 & \text{if } r_i < 3W/10 \end{cases} \quad (4.2)$$

while $N_i^\uparrow(t) = r_i t$; thus if $r_i \geq 3W/10$ we calculate $t_i^\uparrow(n) = n/r_i$, $t_i^\downarrow(n) = n/r_i + 1/2 - W/20r_i$ and

$$\tau_i(n) = 1/2 - W/20r_i \quad (4.3)$$

for all vehicles n since there is no bottleneck on the link; and if $r_i < 3W/10$ we simply have $\tau_i(n) = 1/3$.

Assuming p constant, the inflow rates r_2 and r_3 to links 2 and 3 are related to the diverge by $r_2 = \min\{pW, W/2\}$ and $r_3 = \min\{(1-p)W, W/2\}$. Taking $p \in [1/2, 1]$ by symmetry, we have $r_2 = W/2$ and therefore $\tau_2(n) = 2/5$. Substituting the relation for r_3 into (4.2) we have

$$\tau_3(n) = \begin{cases} \frac{1}{2} \left(1 - \frac{1}{10(1-p)}\right) & \text{if } p \leq 7/10 \\ 1/3 & \text{if } 1/2 \leq p < 7/10 \end{cases} \quad (4.4)$$

Thus, the only equilibrium with constant splitting proportion occurs when $p = 1/2$ and $\tau_2 = \tau_3 = 2/5$.

While this example used a specific piecewise-linear fundamental diagram for concreteness, any such diagram with more than one increasing piece and the same capacity would also produce a unique equilibrium. The more general case is discussed next.

It is natural to ask how a piecewise-linear fundamental diagram should be chosen; one compelling feature of the triangular diagram is its economy of parameters, being completely specified by free-flow speed, capacity, and jam density. One approach is to calibrate directly to traffic observations, but acquiring enough data to accurately calibrate these diagrams for each link in a large roadway network may be prohibitive. Perhaps a simpler idea is to perturb the triangular diagram slightly, so the piecewise-linear diagram retains its general character, but ensures unique speeds (even if very close to free-flow) over a larger range of densities, as in Figure 4.3.

While several perturbation schemes can be imagined, it seems natural to require the capacity

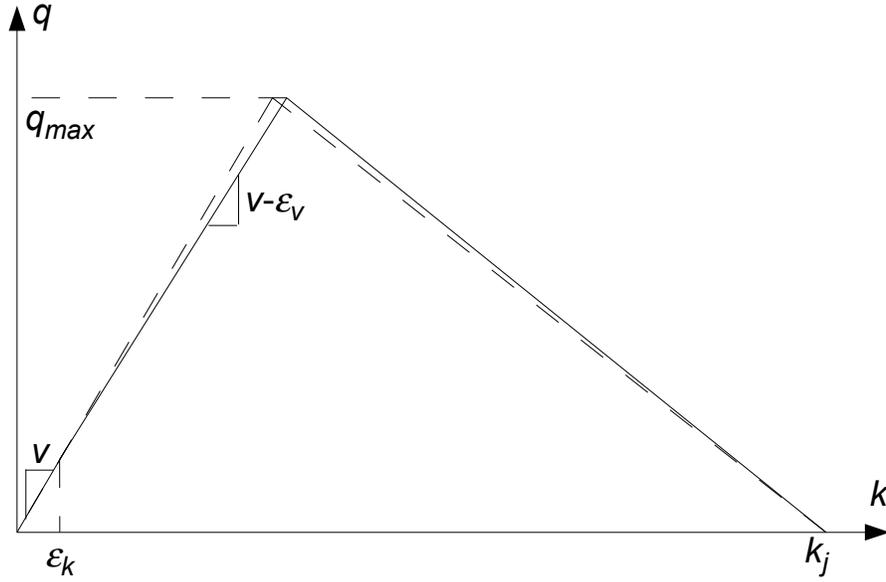


Figure 4.3: Slightly perturbed fundamental diagram approximating triangular shape.

and jam density to remain unchanged while allowing the critical density and backward wave speed to adjust to fit the additional piece, since the former quantities are more often measured in the field or calculated using procedures such as that in the HCM. Such a scheme can be implemented as follows, using several small positive constants (denoted ϵ with appropriate subscripts):

1. For $k \in [0, \epsilon_k]$, $q(k) = vk$, that is, for a small initial portion the fundamental diagram is unchanged.
2. For $k \in [\epsilon_k, k_c + \epsilon_c]$, $q(k) = v\epsilon_k + (v - \epsilon_v)(k - \epsilon_k)$ where ϵ_c is chosen so that $q(k_c + \epsilon_c) = \bar{q}$, that is, on the second piece the flow rises at rate slightly less than free-flow, and continues until capacity is reached.
3. For $k \in [k_c + \epsilon_c, k_j]$, $q(k) = \bar{q}(1 - (k_j - k)/(k_j - k_c - \epsilon_c))$, that is, on the final piece the flow decreases linearly to zero.

Geometric calculations show that the new critical density $k_c + \epsilon_c$ is $(\bar{q} - \epsilon_k \epsilon_v)/(v - \epsilon_v)$, as compared to the triangular critical density $k_c = \bar{q}/v$, and that the new backward wave speed is $\bar{q}(v - \epsilon_v)/(k_j v - \bar{q} + \epsilon_v k_j + \epsilon_k \epsilon_v)$, as compared to the triangular backward wave speed $\bar{q}v/(k_j v - \bar{q})$, and both of these differences shrink to zero as ϵ_v and ϵ_k grow small.

With these fundamental diagrams, the range of demand values producing infinitely many

equilibria is greatly decreased. Solving the LWR model to obtain travel times as a function of link inflow rates yields

$$\tau = \begin{cases} 1/v & \text{if } r \leq v\epsilon_k \\ \frac{r - \epsilon_k\epsilon_v}{r(v - \epsilon_v)} & \text{otherwise} \end{cases} \quad (4.5)$$

Assuming $p \in (0, 1)$ (that is, both links 2 and 3 are used), then $\tau_2 = \tau_3$. If $\tau_2 = \tau_3 = 1/v$, then both links are at free flow, and we must have $pD = r_2 \leq v\epsilon_k$ and $(1-p)D = r_3 \leq v\epsilon_k$. Equivalently, we must have $p \leq v\epsilon_k/D$ and $p \geq 1 - v\epsilon_k/D$, and any p satisfying both inequalities is a user equilibrium. It is impossible to satisfy both inequalities unless $v\epsilon_k/D \geq 1/2$, or $D \leq 2v\epsilon_k$, so unless the demand is quite small, this case will not apply.

If $\tau_2 = \tau_3 > 1/v$, then

$$\frac{r_2 - \epsilon_k\epsilon_v}{r_2(v - \epsilon_v)} = \frac{r_3 - \epsilon_k\epsilon_v}{r_3(v - \epsilon_v)} \quad (4.6)$$

or $r_2 = r_3$ since both the left-hand side and right-hand side are strictly decreasing in their arguments. Therefore the unique equilibrium is $p = 1/2$.

Finally, the cases $p = 0$ and $p = 1$ can be discarded unless $D \leq v\epsilon_k$, in which case any solution is a user equilibrium. Otherwise, $p = 0$ implies $\tau_2 > \tau_3 = 1$ and $p = 1$ implies $1/v = \tau_2 < \tau_3$, neither of which satisfies the Wardrop condition.

To summarize: if $W \leq v\epsilon_k$, all solutions are user equilibria; if $v\epsilon_k \leq W \leq 2v\epsilon_k$, any $p \in [1 - v\epsilon_k/W, v\epsilon_k/W]$ is a user equilibrium; and if $W \geq 2v\epsilon_k$, the unique equilibrium is $p = 1/2$. (Figure 4.4) Furthermore, the maximum possible difference in travel time between the triangular fundamental diagram and perturbed fundamental diagram is

$$\frac{1}{v} - \frac{\bar{q} - \epsilon_k\epsilon_v}{\bar{q}(v - \epsilon_k)} = \frac{1}{v} - \frac{1}{v} \left(1 - \frac{\epsilon_k\epsilon_v}{\bar{q}} \right) \frac{1}{1 - \epsilon_k/v} \quad (4.7)$$

$$= \frac{1}{v} - \frac{1}{v} \left(1 - \frac{\epsilon_k\epsilon_v}{\bar{q}} \right) \left(1 + \frac{\epsilon_k}{v} + \frac{\epsilon_k^2}{v^2} + \dots \right) \quad (4.8)$$

$$= \frac{1}{v} - \frac{1}{v} \left(1 + \frac{\epsilon_k}{v} + \frac{\epsilon_k^2}{v^2} + \dots \right) + \frac{\epsilon_k\epsilon_v}{\bar{q}v} \left(1 + \frac{\epsilon_k}{v} + \frac{\epsilon_k^2}{v^2} + \dots \right) \quad (4.9)$$

$$= \left(\frac{\epsilon_k\epsilon_v}{\bar{q}v} - \frac{\epsilon_k}{v^2} \right) \left(1 + \frac{\epsilon_k}{v} + \frac{\epsilon_k^2}{v^2} + \dots \right) \quad (4.10)$$

using the formula for the sum of a geometric series. As $\epsilon_k \rightarrow 0$, the terms $\frac{\epsilon_k}{v}, \frac{\epsilon_k^2}{v^2}, \dots$ shrink to zero, and the change in travel times is $O(\epsilon_k\epsilon_v)$. Thus, the change in travel times and the range of inflow rates W producing multiple equilibria can be made arbitrarily small by choosing small values of ϵ_k and ϵ_v .

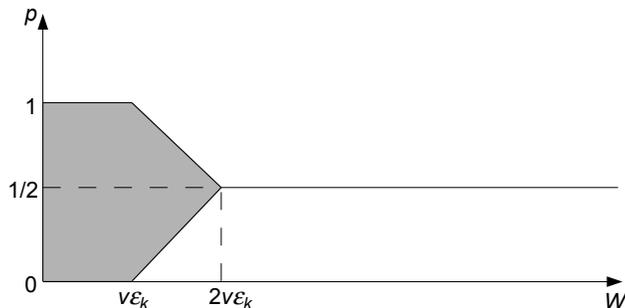


Figure 4.4: Range of user equilibria as a function of inflow rate W .

4.2 Conclusion

We discussed how diverge and traffic flow models can interact to produce counterintuitive phenomena, as demonstrated by an example in which every feasible assignment is a user equilibrium even though the travel times vary widely, and in which the “price of anarchy” can be made arbitrarily high either by increasing the demand or decreasing the link capacities. Our opinion is that this phenomenon is a modeling artifact, rather than something observable in the field. Nevertheless, it poses serious challenges for dynamic traffic assignment algorithms in identifying the “correct” equilibrium solution, and through spillback mechanisms, can propagate unrealistic congestion throughout a network. Simple approaches (such as those based on generating an initial set of paths) may fail to overcome this difficulty, particularly in grid networks.

However, modifying the fundamental diagram by including a second uncongested linear piece can remedy or limit the effects of this phenomenon without compromising the efficiency of LWR solution methods, even when the resulting fundamental diagrams are only slightly perturbed from a triangular one. Further research is needed to quantify the extent to which this phenomenon occurs in larger networks, to investigate other means of resolving or mitigating this modeling issue, and to more generally explore dynamic traffic assignment problems with multiple equilibria.

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